CONVERGENCE OF FREELY DECOMPOSABLE KLEINIAN GROUPS

INKANG KIM, CYRIL LECUIRE AND KEN'ICHI OHSHIKA

Abstract. We prove that for a compact hyperbolic 3-manifold with boundary, a sequence of convex cocompact hyperbolic metrics, whose conformal structures at infinity diverge to a projective lamination which is doubly incompressible, has a compact closure in the deformation space. As a consequence we solve Thurston's conjecture on convergence of function groups affirmatively.

1. Introduction

It is one of the most important topics in the theory of Kleinian group to study the topological structure of deformation spaces of Kleinian groups. For such studies, it is indispensable to guarantee that under some conditions, sequences of Kleinian groups converge inside the deformation spaces. In other words, we need a sufficient condition for a given sequence of Kleinian groups to converge algebraically, which should be as general as possible. In particular, such a sufficient condition for quasi-conformal deformations of geometrically finite Kleinian groups interests us most. The quasi-conformal deformation space of a geometrically finite Kleinian group G is well understood by virtue of the work of Ahlfors, Bers, Kra, Marden and Sullivan. To put it more concretely, for a geometrically finite Kleinian group G, it is known that there is a ramified covering map from the Teichmüller space of Ω_G/G to the quasi-conformal deformation space of G, where Ω_G denotes the region of discontinuity of G. Therefore, the sufficient condition for the quasi-conformal deformation space should be expressed in terms of sequences of the Teichmüller spaces.

The first example of such a sufficient condition for convergence is the result of Bers in [Ber], which shows that the space of quasi-Fuchsian groups lying on the Bers slice is relatively compact. On the other hand, in the process of proving the uniformisation theorem for Haken manifolds, Thurston proved the double limit theorem for quasi-Fuchsian groups and the compactness of deformation spaces for acylindrical manifolds, in [Th1] and [Th2] respectively. These are generalised to give a convergence theorem for general freely

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indecomposable Kleinian groups in Ohshika [Oh1] and [Oh2]. The convergence in the deformation spaces for freely decomposable groups is more complicated and is harder to understand.

In [ThB], Thurston listed questions concerning Kleinian groups and 3-manifolds. One of them asks how one can generalise the double limit theorem to the setting of Schottky groups. This question was made into a more concrete conjecture using the notion of Masur domain, and then was generalised to function groups. Masur introduced in [Ma] an open set in the projective lamination space of the boundary of a handlebody on which the mapping class group of the handlebody acts properly discontinuously. This open set is what we call the Masur domain nowadays. This notion is generalised by Otal [Ot1] to the exterior boundary of a compression body. Thurston's conjecture is paraphrased as follows: For a sequence in the Teichmüller space converging in the Thurston compactification to a projective lamination lying in the Masur domain of the exterior boundary of a compression body M, the corresponding sequence of convex cocompact representations in $AH(M)$ would converge after passing to a subsequence. Otal in [Ot2] first proved that Thurston's conjecture is true for rank-2 Schottky space provided that the limit lamination is arational, that is, any component of its complement is simply connected. Canary in [Ca] proved the conjecture for some special sequences in Schottky space. Ohshika in [Oh4] proved the conjecture for function groups which are isomorphic to the free products of two surface groups under the same assumption that the limit lamination is arational. The strongest result in this direction under the same assumption on the limit lamination was given by Kleineidam and Souto in [KlS] without any other assumption on compression bodies. Our main theorem, Theorem 1 yields a proof of this conjecture of Thurston in full generality without any extra assumption and generalises it to a slightly larger set than the Masur domain.

We need to introduce some notions and notations to state our main theorem. Consider a compact irreducible atoroidal 3-manifold M with boundary. By Thurston's uniformisation theorem for atoroidal Haken manifolds, there is a representation $\rho_0 : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ with the following properties : $\rho_0(\pi_1(M))$ is geometrically finite, $\mathbb{H}^3/\rho_0(\pi_1(M))$ is homeomorphic to $\mathrm{Int}(M)$, and any maximal parabolic subgroup of $\rho_0(\pi_1(M))$ is an Abelian group of rank 2. Such a representation is said to *uniformise* M. Any quasi-conformal deformation of ρ_0 also uniformises M. By the Ahlfors-Bers theory, the space $QH(\rho_0)$ of quasi-conformal deformations of ρ_0 up to conjugacy by elements of $\mathrm{Isom}(\mathbb{H}^3)$ is parametrised by the Teichmüller space of the boundary of M. More precisely, there is a (possibly ramified) covering map, called the *Ahlfors-Bers map* $\mathcal{T}(\partial_{\chi<0} M) \to QH(\rho_0)$ whose covering transformation group is the group of isotopy classes of diffeomorphisms of M which are homotopic to the identity.

The space $QH(\rho_0)$ is a subspace of the *deformation space* $AH(M)$. This deformation space $AH(M)$ is the space of discrete faithful representations ρ :

$\pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ up to conjugacy by elements of $PSL_2(\mathbb{C})$. It is endowed with the compact-open topology which is also called an *algebraic topology.* In the main theorem, we shall consider sequences of representations given by sequences in the Teichmüller space whose images under the Ahlfors-Bers map diverge in $QH(\rho_0)$ and give a sufficient condition for their convergence in $AH(M)$.

Thurston introduced in [Th3] the notion of *doubly incompressible curves.* This can be extended to measured geodesic laminations in the following way.

We say that a measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ is *doubly incompressible* if and only if

- $\exists \eta > 0$ such that $i(\lambda, \partial E) > \eta$ for any essential annulus, Möbius band or disc E.

We denote by $\mathcal{D}(M) \subset \mathcal{ML}(\partial M)$ the set of doubly incompressible measured geodesic laminations and by $\mathcal{PD}(M)$ its projection in the projective lamination space $\mathcal{PML}(\partial M)$. It is not hard to see that $\mathcal{D}(M)$ contains the Masur domain (see [Le2]).

Now we can state our main theorem.

Theorem 1. *Let M be a compact irreducible atoroidal 3-manifold with boundary, and $\rho_0 : \pi_1(M) \to PSL(2, \mathbb{C})$ a geometrically finite representation that uniformises M. Let (m_n) be a sequence in the Teichmüller space $\mathcal{T}(\partial M)$ which converges in the Thurston compactification to a projective measured lamination $[\lambda]$ contained in $\mathcal{PD}(M)$. Let $q : \mathcal{T}(\partial M) \to QH(\rho_0)$ be the Ahlfors-Bers map, and suppose that $(\rho_n : \pi_1(M) \to G_n \subset PSL(2, \mathbb{C}))$ is a sequence of discrete faithful representations corresponding to $q(m_n)$. Then passing to a subsequence, (ρ_n) converges in $AH(M)$.*

The proof of Theorem 1 proceeds as follows. By Theorems of Thurston [Th2] and Canary [Ca], the convergence of m_n to $[\lambda]$ implies that there is a sequence of weighted multi-curves $\lambda_n \in \mathcal{ML}(\partial M)$ such that $l_{\rho_n}(\lambda_n)$ tends to 0 and that the sequence (λ_n) converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination whose projective class is $[\lambda]$. Since λ lies in $\mathcal{D}(M)$, the result comes from the following theorem whose proof occupies the main part of this paper.

Theorem 2. *Let $(\rho_n : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3))$ be a sequence of discrete faithful representations that uniformise M and let $(\lambda_n) \subset \mathcal{ML}(\partial M)$ be a sequence of measured geodesic laminations such that $(l_{\rho_n}(\lambda_n))$ is a bounded sequence and that (λ_n) converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination $\lambda \in \mathcal{D}(M)$. Then the sequence (ρ_n) has a compact closure in $AH(M)$, namely, any subsequence contains an algebraically convergent subsequence.*

It should be noted that our result here is closely related to the Bers-Thurston density conjecture. This conjecture states that every finitely generated Kleinian group would be contained in the boundary of the quasi-conformal deformation space of geometrically finite Kleinian groups without rank-1 maximal parabolic subgroups. The conjecture was proved by

combining the resolution of the tameness conjecture by Agol and Calegari-Gabai, the ending lamination conjecture by Brock-Canary-Minsky, and from some convergence theorems due to Thurston, Ohshika, Kleineidam-Souto and Lecuire, together with some topological argument due to Ohshika and Namazi-Souto(See [Ag], [CG], [Th3], [Oh4], [KlS], [Le2], [Min], [OhP] and [Na]). Although the convergence theorems less general than ours were sufficient for this proof, we can simplify some of the argument there using our general result in this paper.

2. Preliminaries

2.1. **Deformation space.** Let G be a finitely generated torsion-free Kleinian group, namely a (torsion-free and finitely generated) discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. We denote by $(\mathbb{H}^3/G)_0$ the non-cuspidal part of $(\mathbb{H}^3/G)$, that is, the complement of neighbourhoods of the cusps. We call an end of $(\mathbb{H}^3/G)_0$ *geometrically finite* when it has a neighbourhood intersecting no closed geodesics, and otherwise *geometrically infinite*. The Kleinian group G is geometrically finite if and only if every end of $(\mathbb{H}^3/G)_0$ is geometrically finite. Let $\Omega_G \subset S^2_\infty$ denote the region of discontinuity for the action of G on the Riemann sphere S^2_∞. Each geometrically finite end corresponds to a component Σ of Ω_G/G in such a way that a neighbourhood of the end is compactified in the Kleinian manifold $(\mathbb{H}^3 \cup \Omega_G)/G$ by adding Σ.

Let M be a compact irreducible atoroidal manifold. Let $AH(M)$ denote the set of faithful discrete representations from $\pi_1(M)$ to $PSL(2,\mathbb{C})$ modulo conjugacy. We endow $AH(M)$ with the topology induced from the representation space. We say that a representation $\rho \in AH(M)$ has minimal parabolics if all of its maximal parabolic subgroups have rank 2. The subspace of $AH(M)$ consisting of geometrically finite representations with minimal parabolics is denoted by $CC(M)$. This space $CC(M)$ may contain several connected components. The component consisting of representations ρ for which there is a homeomorphism from $\mathrm{Int}(M)$ to $\mathbb{H}^3/\rho(\pi_1(M))$ is denoted by $CC_0(M)$. The representations in $CC_0(M)$ are said to uniformise M.

For a Kleinian group G, if there is a quasi-conformal automorphism f of S^2_∞ such that fGf^{-1} is again a Kleinian group, then this group fGf^{-1} is said to be a *quasi-conformal deformation* of G. Quasi-conformal deformations preserve the parabolic subgroups. Therefore a quasi-conformal deformation of a Kleinian group with minimal parabolics has minimal parabolics. Furthermore quasi-conformal deformations of a geometrically finite group are geometrically finite. By the theory of Ahlfors-Bers, there is a ramified covering map from $\mathcal{T}(\Omega_G/G)$ to the space of quasi-conformal deformations of G modulo conjugacy. Let $\partial_{\chi<0}M$ be the union of the components of ∂M which have negative Euler characteristics. For $G \in CC_0(M)$, there is a natural identification of Ω_G/G with $\partial_{\chi<0}M$. The covering map $\mathcal{T}(\partial_{\chi<0}M) \to CC_0(M)$ is called the *Ahlfors-Bers map.*

2.2. **$\mathbb{R}$-trees.** An $\mathbb{R}$-tree $\mathcal{T}$ is a geodesic metric space in which any two points x, y can be joined by a unique simple arc. Let G be a group acting by isometries on an $\mathbb{R}$-tree $\mathcal{T}$. The action is *minimal* if there is no proper invariant subtree and *small* if the stabilizer of any non-degenerate arc is virtually Abelian.

Morgan and Shalen [MoS1] made use of $\mathbb{R}$-trees to compactify deformation spaces. They used algebraic methods involving valuations, while the same result has been obtained in Paulin [Pa] and Bestvina [Bes] using a more geometrical approach. In this paper we shall adopt the point of view of Kapovich-Leeb [KaL] (see also [Ka, chapters 9 and 10]). Let $(\rho_n \subset CC_0(M))$ be a sequence of representations such that no subsequence of (ρ_n) converges algebraically. Let $\Gamma \subset \pi_1(M)$ be a set of generators and let $\tilde{x}_n \subset \mathbb{H}^3$ be a point realising the minimum ϵ_n^{-1} on $\mathbb{H}^3$ of the function $\max\{d(\tilde{x}, \rho_n(a)(\tilde{x})), a \in \Gamma\}$ (see for example [Pa] for the existence of such a point). Since no subsequence of (ρ_n) converges algebraically, (ϵ_n^{-1}) tends to ∞. Choose a non-principal ultra-filter ω and denote by $\epsilon_n\mathbb{H}^3$ the hyperbolic space $\mathbb{H}^3$ with the hyperbolic metric rescaled by ϵ_n. The ultra-limit $(X_\omega, x) = \omega - \lim(\epsilon_n\mathbb{H}^3, x_n)$ of the sequence of rescaled spaces is defined as follows. Let $\Pi_n(\epsilon_n\mathbb{H}^3)$ be the infinite product of the spaces $(\epsilon_n\mathbb{H}^3)$. We define a pseudo-distance d_ω on $\Pi_n(\epsilon_n\mathbb{H}^3)$ by setting

$$d_\omega(y, z) = \omega - \lim d_{\epsilon_n\mathbb{H}^3}(\tilde{y}_n, \tilde{z}_n)$$

for any two points $y = (\tilde{y}_n)$ and $z = (\tilde{z}_n)$ lying in $\Pi_n(\epsilon_n\mathbb{H}^3)$.

This function d_ω is a pseudo-distance in $\Pi_n(\epsilon_n\mathbb{H}^3)$ with values in $[0, \infty]$ and we set $(X_\omega, d_\omega) = (\Pi_n(\epsilon_n\mathbb{H}^3), d_\omega)/\sim$ where we identify points with zero d_ω-distance. Let $x = (\tilde{x}_n)$ denote the sequence of points $\tilde{x}_n$ defined above. The metric space (X_ω, x) is the set of points of (X_ω) with a finite distance from x. This metric space is an $\mathbb{R}$-tree (cf. [KaL]). The action of $\rho_n(\pi_1(M))$ on $\epsilon_n\mathbb{H}^3$ gives rise to an action of $\pi_1(M)$ on (X_ω, x) by isometries. This action is small (cf. [KaL]). Let $\mathcal{T}$ be the minimal invariant subtree of X_ω under this action. We say that (ρ_n) tends to the action of $\pi_1(M)$ on $\mathcal{T}$ with respect to ω. For $c \in \pi_1(M)$ let us denote by $\delta_{\mathcal{T}}(c)$ the minimal translation distance of c on $\mathcal{T}$. Then we have $\delta_{\mathcal{T}}(c) = \lim \epsilon_n^{-1} l_{\rho_n}(c)$, where $l_{\rho_n}(c)$ is the length in $\mathbb{H}^3/\rho_n(\pi_1(M))$ of the closed geodesic in the free homotopy class of c.

2.3. **Geodesic laminations.** A *geodesic lamination* L on a complete hyperbolic surface S is a compact set which is a disjoint union of complete embedded geodesics called *leaves*. It is a classical fact that this definition is independent of a chosen hyperbolic metric on S (see [Ot3] for example). For a connected geodesic lamination L which is not a simple closed curve we denote by $\bar{S}(L)$ the smallest surface with compact geodesic boundary containing L. Inside $\bar{S}(L)$ there are finitely many closed geodesics (including the components of $\partial\bar{S}(L)$) disjoint from L. These closed geodesics do not intersect each other (cf. [Le1]) and we denote by $\partial'\bar{S}(L) \supset \partial\bar{S}(L)$ their

disjoint union. Removing a small tubular neighbourhood of $\partial'\bar{S}(L)$ from $\bar{S}(L)$ we get an open surface $S(L)$. We call $S(L)$ the *surface embraced* by the geodesic lamination L and $\partial'\bar{S}(L)$ the *effective boundary* of $S(L)$. If L is a simple closed curve, we define $S(L)$ to be an annular neighbourhood of L and we take $\partial'\bar{S}(L) = L$. When L is not connected, $S(L)$ is the disjoint union of the surfaces embraced by the connected components of L.

We say that two geodesic measured laminations L and L' intersect transversely if *at least* one leaf of L intersects a leaf of L' transversely.

A *measured geodesic lamination* λ is a geodesic lamination $|\lambda|$ together with a transverse measure. We always assume that λ has $|\lambda|$ as its support. We denote by $\mathcal{ML}(S)$ the space of measured geodesic laminations on S endowed with the weak-$*$ topology. To simplify the notations, we write $\mathcal{ML}(\partial M)$ instead of $\mathcal{ML}(\partial_{\chi<0} M)$ for a compact 3-manifold M with boundary. The projective lamination space $\mathcal{PML}(\partial M)$ is defined to be $(\mathcal{ML}(\partial M) - \{0\})/\mathbb{R}_+^*$ where 0 stands for the measured lamination with empty support. It should be noted that $\mathcal{ML}(\partial M)$ contains measured laminations whose restriction to some component of ∂M is empty. The Teichmüller space $\mathcal{T}(\partial M)$ denotes similarly $\mathcal{T}(\partial_{\chi<0} M)$. The boundary of the Thurston compactification of $\mathcal{T}(\partial M)$ is equal to $\mathcal{PML}(\partial M)$.

2.4. **Train tracks and their realisations.** A *train track* τ in a hyperbolic surface S is a union of finitely many rectangles which meet each other only along non-degenerate segments contained in their vertical sides. The rectangles are called *branches*, and they are foliated by vertical segments called *ties*. A connected component of the intersection of the branches is called a *switch*. The branches are also foliated by horizontal segments, and a smooth arc which is a union of horizontal segments is called a *rail* or a *train route*. A geodesic lamination is *carried* by τ if it lies in τ in such a way that the leaves are transverse to the ties (see [Bo] or [Ot3] for more details about train tracks).

When τ is a train track and λ is a measured geodesic lamination whose support is carried by λ, we say that λ is carried by τ. For a branch b of τ, we define the number $\lambda(b)$ to be the λ-measure of a tie of b. This number does not depend on the choice of a tie in b.

Consider an action of $\pi_1(S)$ on an $\mathbb{R}$-tree $\mathcal{T}$ by isometries. A measured lamination λ is said to be *realised* in $\mathcal{T}$ if there is an equivariant map (with respect to the action of $\pi_1(S)$) $\phi : \mathbb{H}^2 \to \mathcal{T}$ such that the restriction of ϕ to any lift of a leaf of λ in $\mathbb{H}^2$ is monotonic and non-constant. A train track $\tau \in S$ is said to be realised in $\mathcal{T}$ if there is an equivariant map $\phi : \mathbb{H}^2 \to \mathcal{T}$ which maps each lift of a branch of τ to a non-trivial geodesic segment on $\mathcal{T}$ in such a way that each rail is mapped injectively, and that each tie collapses to a point. By [Ot3], λ is realised in $\mathcal{T}$ if and only if λ is carried by a train track τ which is realised in $\mathcal{T}$.

We say that a measured lamination λ is *collapsed* by ϕ when there is a train track τ carrying λ such that every component of $\tilde{\tau}$, the preimage of τ

in $\mathbb{H}^2$, is mapped to a point by ϕ. It is straightforward that there exists an equivariant map collapsing λ if and only if the action of $i_*(\pi_1(S(\lambda))$ on $\mathcal{T}$ has a global fixed point.

2.5. **Compactification of $\tilde{M}$.** We denote by $\tilde{M}$ the universal covering of M and by $p : \tilde{M} \to M$ the covering projection. We compactify $\tilde{M}$ in the following way : endow M with a geometrically finite hyperbolic metric σ with minimal parabolics and let us denote by $N(\sigma)^{\text{thick}}$ the complement in the convex core $N(\sigma)$ of ϵ-thin neighbourhood of the cusps of σ for some ϵ smaller than the Margulis constant. Let us choose an isometry between the interior of $\tilde{M}$ and $\mathbb{H}^3$. Now we can consider $\tilde{N}(\sigma)^{\text{thick}}$ as a closed subset of $\mathbb{H}^3$. Since σ is geometrically finite, there is a natural homeomorphism between M and $N(\sigma)^{\text{thick}}$. Therefore we can regard $\tilde{M}$ as a closed subset of $\mathbb{H}^3$. The compactification $\overline{\tilde{M}}$ of $\tilde{M}$ is the closure of this closed subset in the usual compactification of $\mathbb{H}^3$ by the unit ball. If we replace σ by another geometrically finite metric σ' with minimal parabolics, it follows from results of [Fl] that we get a compactification which is homeomorphic to the one obtained with σ. Therefore this definition is independent of the metric we chose. We call this the Floyd-Gromov compactification of $\tilde{M}$.

A *meridian* is a simple closed curve $c \subset \partial M$ which bounds a disc in M but not on ∂M. A compact surface $\Sigma \subset \partial M$ is incompressible if it contains no meridians. When we consider the closure of a lift of an incompressible surface in $\overline{\tilde{M}}$, we have the following :

Lemma 2.1. *Let $\Sigma \subset \partial_{\chi<0} M$ be a compact connected incompressible surface which does not contain any essential closed curve homotopic into $\partial_{\chi=0} M$. Let $\hat{\Sigma}$ be the universal covering of Σ, which is completed in the usual way to a closed disc $\overline{\hat{\Sigma}}$. Then any lift of Σ to $\partial\tilde{M}$ is a disc whose closure in $\overline{\tilde{M}}$ is homeomorphic to $\overline{\hat{\Sigma}}$ in an equivariant way.*

Proof. This Lemma was proved in [Ot1], see also [Le1, Lemme 2.1]. $\square$

Let $\Sigma \subset \partial M$ be a compact incompressible surface. Johannson and Jaco-Shalen defined a characteristic submanifold W relative to Σ (cf. [Jo] and [JaS]). Notice that such a characteristic manifold was defined for any incompressible surface in the boundary of a Haken manifold. Since we are considering atoroidal M now, such a characteristic submanifold is a disjoint union of essential I-bundles over closed surfaces and essential solid tori. A disjoint union W of essential I-bundles and essential solid tori is a characteristic submanifold if and only if it has the following two properties :

- any essential I-bundle and any essential solid torus in (M, Σ) can be homotoped in W;
- no connected component of W can be homotoped into another connected component of W.

By [Jo] and [JaS], if W and W' are two characteristic submanifolds relative to Σ, then there is a diffeomorphism $\psi : M \to M$ isotopic to the identity relative to $\partial M - \Sigma$ such that $\psi(W) = W'$ and that $\psi(W \cap \Sigma) = W' \cap \Sigma$.

Such a characteristic submanifold can be found by looking only at $\overline{\tilde{M}} - \tilde{M}$.

Proposition 2.2 ([Le1], §2)**.** *Let Σ and $\Sigma' \subset \partial_{\chi<0} M$ be two compact, connected, incompressible surfaces which are disjoint or equal and do not contain any essential closed curve which can be homotoped in $\partial_{\chi=0} M$. Let $\tilde{\Sigma} \subset \partial\tilde{M}$ (resp. $\tilde{\Sigma}'$) be a connected component of the preimage of Σ (resp. Σ') and let $\Gamma \subset \rho(\pi_1(M))$ (resp. Γ') be the stabiliser of $\tilde{\Sigma}$ (resp. Γ').*

Then $\overline{\tilde{\Sigma}} \cap \overline{\tilde{\Sigma}}'$ is either empty or equal to the limit set of $\Gamma \cap \Gamma'$.

In the latter case, if $\Gamma \cap \Gamma'$ is not cyclic, then it is the fundamental group of an I-bundle which is a connected component of a characteristic submanifold of $(M, \Sigma \cup \Sigma')$. If $\Gamma \cap \Gamma'$is cyclic, then it is a finite index subgroup of a solid torus which is a connected component of a characteristic submanifold of $(M, \Sigma \cup \Sigma')$.

2.6. **Geodesic laminations on compressible surfaces.** Let M be a compact 3-manifold with boundary and let $c \subset \partial M$ be a simple closed curve. A *c-wave* is a simple arc k, with $k \cap c = \partial k$ such that there is an arc κ in c with the simple closed curve $k \cup \kappa$ bounding an essential disc in M. In some literature, a c-wave is allowed to intersect c in its interior. A simple innermost argument shows that if there is a c-wave in this generalised sense, there is one in our sense.

Let L be a geodesic lamination on $\partial_{\chi<0} M$ and let $c \subset \partial M$ be a multicurve. In the following, we always assume that simple closed curves or multicurves are geodesics, hence there are no inessential intersection between them or with geodesic laminations. We say that L is in *tight position* with respect to c if L contains no c-waves and if *every* leaf of L intersects c transversely.

The following Claim shows how to use cut-and-paste operations to construct a meridian m such that a given geodesic lamination contains no m-waves.

Claim 2.3. *Let $F \subset \partial M$ be a compressible compact surface and let $\beta \subset F$ be a measured geodesic lamination. Then either β intersects transversely a meridian m and contains no m-waves, or there is a sequence of meridians $(m_i \subset F)$ converging in the Hausdorff topology to a geodesic lamination which does not intersect β transversely.*

Proof. If β intersects no meridians transversely, then, since F is compressible, there is a meridian $m \in F$ such that $i(\beta, m) = 0$. Setting $m_i = m$ for every i, we get the conclusion. Now we assume that β intersects a meridian m transversely. If β contains an m-wave k, let us "cut m along k" to get a new meridian m_1 : let κ be the closure of a connected component of $m - \partial k$ such that $\int_\kappa d\beta \leq \frac{1}{2} i(\beta, m)$, and m_1 the simple closed geodesic in the free

homotopy class of $k \cup \kappa$. We have $i(\beta, m_1) \leq \frac{1}{2} i(\beta, m)$ and m_1 is a meridian. If β does not contains any m_1-waves, we are done. If β contains an m_1-wave k_1, then we cut m_1 along k_1 as above to get a new meridian m_2 with $i(m_2, \beta) \leq \frac{1}{4} i(m, \beta)$. Repeating this operation, we get either a meridian m' such that β contains no m'-waves or a sequence of meridians (m_i) such that $i(m_i, \beta) \longrightarrow 0$. In the latter case, we extract a subsequence such that m_i converges in the Hausdorff topology to some geodesic lamination H. Since $i(m_i, \beta) \longrightarrow 0$, we see that H does not intersect β transversely. □

Let L be a geodesic lamination on $\partial_{\chi<0} M$. A leaf l of L is said to be *homoclinic* if a lift $\tilde{l}$ of l to the universal cover $\tilde{M}$ of M contains two sequences of point $(\tilde{x}_n)$ and $(\tilde{y}_n)$ such that the distance between $\tilde{x}_n$ and $\tilde{y}_n$ in $\tilde{M}$ is uniformly bounded whereas their distance measured on $\tilde{l}$ tends to ∞. By Lemma 2.1 and by [Le1, Affirmation 3.4], an incompressible surface cannot contain a homoclinic leaf.

Homoclinic leaves appear naturally in Hausdorff limits of sequences of meridians. This is illustrated by the following criterion of Casson whose proof can be found in [Ot1] and [Le1, Theorem B.1].

Lemma 2.4. *Let $(m_n \subset \partial M)$ be a sequence of meridians which converges to a geodesic lamination H in the Hausdorff topology. Then H contains a homoclinic leaf.*

A *simple half-geodesic* is an embedded half-line in ∂M whose image is locally geodesic for some hyperbolic metric on $\partial_{\chi<0} M$. Let $\tilde{l}_+ \subset \partial\tilde{M}$ be a half-geodesic and let $\bar{\tilde{l}}_+$ be its closure in the Floyd-Gromov compactification of $\tilde{M}$. We say that $\tilde{l}_+$ *has a well-defined endpoint* if $\bar{\tilde{l}}_+ - \tilde{l}_+$ contains one point. We say that a geodesic $\tilde{l} \subset \partial\tilde{M}$ has two well-defined endpoints if $\tilde{l}$ contains two disjoint half geodesics each having a well-defined endpoint. Notice that we allow the two endpoints to be the same. Two distinct leaves $\tilde{l}_1$ and $\tilde{l}_2$ of a geodesic lamination $\tilde{L} \subset \partial\tilde{M}$ are said to be *biasymptotic* if they both have two well-defined endpoints in $\overline{\tilde{M}}$ and the endpoints of $\tilde{l}_1$ coincide with those of $\tilde{l}_2$. A geodesic lamination $A \subset \partial M$ is said to be *annular* if the preimage of A in $\partial\tilde{M}$ contains a pair of biasymptotic leaves.

Annular laminations appear naturally as limits of sequences of annuli.

Lemma 2.5. *Let $(A_n \subset \partial M)$ be a sequence of (not necessarily embedded) essential annuli or Möbius bands. Suppose that ∂A_n converges in the Hausdorff topology to a geodesic lamination E. Suppose also that there is a measured geodesic lamination α whose support is equal to E such that $i(\alpha, \partial A_n) \longrightarrow 0$. Then either E is annular or $S(E)$ contains a homoclinic leaf which does not intersect E transversely.*

Proof. The main argument of this proof comes from [Le1, Lemme C.2].

Let $p : \tilde{M} \to M$ be the covering projection. Fix some complete hyperbolic metric on $\partial_{\chi<0} M$, and let $S(E)$ be the surface embraced by E.

Assume first that $S(E)$ is incompressible. Let $\tilde{S}(E) \subset \partial\tilde{M}$ be a lift of $S(E)$, and $\tilde{F}$ a fundamental domain for the action of $\pi_1(S(E))$ on $\tilde{S}(E)$. Let $\partial^1\tilde{A}_n$ and $\partial^2\tilde{A}_n$ be the boundary components of a lift of A_n to $\tilde{M}$ such that $\partial^1\tilde{A}_n$ intersects $\tilde{F}$. By extracting a subsequence, we can assume that the sequence $(\partial^1\tilde{A}_n)$ converges to a lift $\tilde{e}$ of a leaf e of E. Since A_n is an annulus or a Möbius band, the geodesic $\partial^2\tilde{A}_n$ has the same ends as $\partial^1\tilde{A}_n$. By [Le1, Affirmation 2.2], for any $\epsilon > 0$, there are only finitely many translates of $\tilde{S}(E)$ whose limit sets have diameters greater than ϵ. Therefore, after extracting a subsequence, all the geodesics $\partial^2\tilde{A}_n$ can be assumed to lie in the same lift $\tilde{S}(E)' \subset \partial\tilde{M}$ of $S(E)$. The ends of $\partial^2\tilde{A}_n$ converge to the ends of $\tilde{e} \subset \tilde{S}(E)$. By Lemma 2.1, a geodesic in $\tilde{S}(E)'$ is uniquely determined by its two ends. Therefore the sequence $(\partial^2\tilde{A}_n)$ converges to the lift $\tilde{e}' \subset \tilde{S}(E)'$ of a leaf e' of E which has the same ends as $\tilde{e}$. If $\tilde{S}(E) = \tilde{S}(E)'$ then $\partial^1\tilde{A}_n$ and $\partial^2\tilde{A}_n$ lie in $\tilde{S}(E)$. This contradicts Lemma 2.1 since $\partial^1 A_n$ and $\partial^2 A_n$ are not homotopic in $S(E)$. Hence we have $\tilde{S}(E) \neq \tilde{S}(E)'$ and $\tilde{e} \neq \tilde{e}'$. It follows that E is an annular lamination.

Let us now consider the case when $S(E)$ is compressible. By Claim 2.3 and the Casson's criterion (Lemma 2.4), either there is a meridian $m \subset S(E)$ such that E does not contain any m-wave, or $S(E)$ contains a homoclinic leaf which does not intersect E transversely. Therefore we have only to consider the case when there is a meridian $m \subset S(E)$ such that E intersects m transversely and contains no m-waves.

We set $\partial^1 A_n = S^1 \times \{0\}$ and $\partial^2 A_n = S^1 \times \{1\}$ where $A_n = S^1 \times I$. We shall first show that E is in tight position with respect to m. For this, we need to show that every leaf of E intersects m. Since E is the Hausdorff limit of ∂A_n, it has one or two connected components. Since there is a measured geodesic lamination whose support is equal to E, each of these components is a minimal lamination. Therefore if a leaf of E intersects m, the same is true for all the leaves lying in the same connected component. Thus if E has only one component, E is in tight position with respect to m.

Now we assume that there are two connected components E^1, E^2 of E. Since E intersects m, we can assume that E^1 intersects m. By extracting a subsequence, we can also assume that E^1 is the Hausdorff limit of some $\partial^i A_n$, say $\partial^1 A_n$ and that E^2 is the Hausdorff limit of $\partial^2 A_n$. Note that $\partial^1 A_n$ contains no m-waves for sufficiently large n (even if we allow an m-wave to have self-intersection). Otherwise taking a limit in the Hausdorff topology of m-waves lying in $\partial^1 A_n$, we get either an m-wave lying in E contradicting the fact that E is in tight position with respect to m, or a sublamination of E^1 not intersecting m, contradicting the fact that E^1 is a minimal lamination intersecting m. It follows from the fact that $\partial^1 A_n$ contains no m-waves that the ends of any lift of $\partial^1 A_n$ to $\tilde{M}$ are separated by a lift of m (compare with [Le1, Affirmation 3.4]). This implies that $\partial^2 A_n$ intersects m. Since E^2 is the Hausdorff limit of $\partial^2 A_n$, we see that E^2 also intersects m.

Thus we have proved that E is in tight position with respect to m whether E has one or two connected components. Let us deduce that there is a uniform upper bound K on the lengths of the connected components of $\partial A_n - m$. Assuming the contrary, we get a sequence of arcs $k_n \subset \partial A_n - m$ whose lengths tend to ∞. Extract a subsequence such that (k_n) converges in the Hausdorff topology. The limit contains a sublamination of E which does not intersect m, contradicting the fact that E is in tight position with respect to m.

Let e be a leaf of E and let $\tilde{e} \subset \tilde{M}$ be a lift of e. Denote by $(\tilde{m}_j)_{j\in\mathbb{Z}}$ the connected components of $p^{-1}(m)$ that $\tilde{e}$ intersects (in this order). Let $\tilde{e}[-t,t]$ be the segment of $\tilde{e}$ connecting $\tilde{m}_{-t}$ to $\tilde{m}_t$. By assumption, ∂A_n converges to E in the Hausdorff topology. Therefore, for sufficiently large n, a connected component $\partial^1\tilde{A}_n$ of $p^{-1}(\partial A_n)$ intersects $\tilde{m}_{-t}$ and $\tilde{m}_t$ and $\partial^1\tilde{A}_n[-t,t]$ converges to $\tilde{e}[-t,t]$ in the Hausdorff topology. Let $\tilde{A}_n$ be the lift of A_n whose boundary contain $\partial^1\tilde{A}_n$. As we have seen above, ∂A_n does not contain any m-wave for sufficiently large n. Therefore both $\tilde{m}_t$ and $\tilde{m}_{-t}$ separate the ends of $\partial^1\tilde{A}_n$ (cf. [Le1, Affirmation 3.4]). It follows that $\tilde{m}_t$ and $\tilde{m}_{-t}$ intersect $\partial^2\tilde{A}_n$. Let $\partial^2\tilde{A}_n[-t,t]$ be the segment of $\partial^2\tilde{A}_n$ joining $\tilde{m}_t$ to $\tilde{m}_{-t}$. The length of $\partial^2\tilde{A}_n[-t,t]$ is less than $2Kt$ where K is the constant found in the previous paragraph. After extracting a subsequence, $(\partial^2\tilde{A}_n[-t,t])$ converges in the Hausdorff topology to an arc $\tilde{e}'[-t,t] \subset p^{-1}(E)$. Letting t tend to ∞, we conclude that $\tilde{e}$ and $\tilde{e}'$ have the same ends.

If $\tilde{e} = \tilde{e}'$, then $\partial^1\tilde{A}_n$ and $\partial^2\tilde{A}_n$ intersect an ϵ-neighbourhood of $\tilde{e}$ for sufficiently large n. Hence there are segments $\tilde{I}_n \subset \partial\tilde{M}$ joining the two components of $\partial\tilde{A}_n$ with their projections I_n such that $\int_{I_n} d\alpha$ tends to 0. Since $\tilde{I}_n$ is homotopic to an arc on $\tilde{A}_n$ joining two boundary components, I_n is homotopic to an essential arc on A_n joining two boundary components. If we cut ∂A_n along ∂I_n and glue two copies of I_n, we get a closed curve which bounds a disc D_n (not necessarily embedded). By assumption, $i(\alpha, \partial A_n)$ and $\int_{I_n} d\alpha$ tend to 0. Therefore $i(\alpha, \partial D_n)$ tends to 0. By the Loop Theorem (cf. [He]), there is an embedded disc D'_n which is not parallel to ∂M such that $i(\alpha, \partial D'_n) \longrightarrow 0$. Extracting a subsequence such that $(\partial D'_n)$ converges in the Hausdorff topology, we get a geodesic lamination $H \subset S(E)$ which does not intersects $E = |\alpha|$ transversely. By Casson's criterion (Lemma 2.4), H contains a homoclinic leaf.

When $\tilde{e} \neq \tilde{e}'$, by definition, E is annular. □

As in the introduction, we say that a measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ is *doubly incompressible* if and only if :

- $\exists \eta > 0$ such that $i(\lambda, \partial E) > \eta$ for every essential annulus, Möbius band or disc E.

We denote by $\mathcal{D}(M) \subset \mathcal{ML}(\partial M)$ the set of doubly incompressible measured geodesic laminations and by $\mathcal{PD}(M)$ its projection in the space $\mathcal{PML}(\partial M)$

of projective measured laminations.

Some properties of this set $\mathcal{D}(M)$ are discussed in [Le2]. In particular, we can deduce the following from [Le1].

Lemma 2.6. *Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination and $l_+, l_- \subset \partial M$ two simple half-geodesics which do not intersect $|\lambda|$ transversely. Then any lift of l_+ (resp. l_-) to $\tilde{M}$ has a well-defined endpoint.*

Furthermore, if a lift of l_+ has the same endpoint as a lift of l_- then l_+ and l_- are asymptotic on ∂M.

Proof. The first property, namely that any lift of l_+ (resp. l_-) to $\tilde{M}$ has a well-defined endpoint, can be deduced from the proofs of [Le1, Lemme 3.1] and [Le1, Lemme 3.3].

The proof of the second property, namely that if a lift of l_+ has the same endpoint as a lift of l_- then l_+ and l_- are asymptotic on ∂M, can be found in the paragraph (ii) in the proof of [Le1, Lemme C3]. □

From [Le2], we also get the following :

Lemma 2.7. *Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination, A an annular geodesic lamination which is the Hausdorff limit of a sequence of multi-curves, and h a homoclinic leaf (of some geodesic lamination). Then the support $|\lambda|$ of λ intersects both A and h transversely.*

Proof. When M is not a genus-2 handlebody, this is [Le2, Lemma 3.5]. The case when M is a genus 2 handlebody is discussed in [Le2] in the remark following [Le2, Lemma 3.5]. □

Combining Claim 2.3, Casson's criterion (Lemma 2.4) and Lemma 2.7, we get the following :

Lemma 2.8. *Let $\lambda \in \mathcal{D}(M)$, and let $S \subset \partial M$ be a compressible surface. Then there is a meridian m in S such that S does not contain any m-waves disjoint from $|\lambda|$.*

2.7. **Some notations.** When λ is a geodesic measured lamination, we denote by $|\lambda|$ the support of λ. For an arc k whose intersections with $|\lambda|$ are transverse, we will denote by $\int_k d\lambda$ the λ-measure of k.

Let (u_n) and (v_n) be two sequences of non-negative real numbers. We say that u_n is $o(v_n)$ if for any ϵ there is $N(\epsilon)$ such that for $n \geq N(\epsilon)$, we have $u_n \leq \epsilon v_n$. We also write $u_n = o(v_n)$.

We will say that u_n is $O(v_n)$ if there are $K, N > 0$ such that for $n \geq N$, we have $u_n \leq K v_n$.

We say that u_n is $\Theta(v_n)$ if u_n is $O(v_n)$ and v_n is $O(u_n)$.

3. Realisations of doubly incompressible laminations

Let $\pi_1(M) \curvearrowright \mathcal{T}$ be a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree for a compact irreducible atoroidal 3-manifold M. Let S be a connected component of $\partial_{\chi<0} M$. Using the map $i_* : \pi_1(S) \to \pi_1(M)$ induced by the inclusion,

we get an action of $\pi_1(S)$ on $\mathcal{T}$. Therefore, if $\lambda \in \mathcal{ML}(S)$ is a measured geodesic lamination, it makes sense to ask whether or not it is realised in $\mathcal{T}$. In this section, we shall discuss this question for the connected components of a measured lamination lying in $\mathcal{D}(M)$.

Lemma 3.1. *Let $\pi_1(M) \curvearrowright \mathcal{T}$ be a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree. Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination, and α a minimal sublamination of λ. Then one of the following holds:*

- *the measured lamination α is realised in $\mathcal{T}$;*
- *there is a sequence of train tracks θ_i each of which minimally carries α and has the following properties:*
  - *θ_i has only one switch κ_i and $\kappa_i \supset \kappa_{i+1}$ for every i.*
  - *There are a point $p \in \mathcal{T}$, a sequence $\eta_i \to 0$, and a sequence of $\pi_1(S)$-equivariant maps $\phi_i : \mathbb{H}^2 \to \mathcal{T}$ such that ϕ_i maps every branch of the preimage of θ_i to a geodesic segment (which may be a point) with length smaller than η_i and such that a preimage of κ_i is mapped to p under ϕ_i. Note that the number of branches of θ_i is uniformly bounded.*

Furthermore there is at least one component of λ which is realised in $\mathcal{T}$.

Proof. Notice that when α is collapsed in $\mathcal{T}$, it is easy to see that we are in the second situation.

Let us first assume that α is a simple closed curve. If $\delta_{\mathcal{T}}(\alpha) = 0$ then α is collapsed in $\mathcal{T}$, if $\delta_{\mathcal{T}}(\alpha) \neq 0$, then α is realised in $\mathcal{T}$.

Now assume that α is not a simple closed curve. We cut M along a maximal family of essential discs disjoint from $S(\alpha)$. We denote by N the connected component of the resulting manifold that contains α on its boundary. The surface $\partial N - S(\alpha)$ is incompressible in N. Let $\mathcal{T}_N$ be the minimal subtree of $\mathcal{T}$ that is invariant under the action of $\pi_1(N)$ regarded as a subgroup of $\pi_1(M)$. If $\mathcal{T}_N$ is trivial, then α is collapsed. Therefore, from now on we assume that $\mathcal{T}_N$ is not trivial. Let us denote by F the connected component of ∂N containing $S(\alpha)$. Since λ lies in $\mathcal{D}(M)$, every component of $\partial S(\alpha)$ is essential in M; hence $S(\alpha)$ is incompressible on F. In particular, the measured lamination α can be regarded as lying on F. Using the map $i_* : F \to N$ induced by the inclusion, we get an action of $\pi_1(F)$ on $\mathcal{T}_N$. Let $\mathcal{T}_F$ be the minimal subtree of $\mathcal{T}_N$ that is invariant under the action of $\pi_1(F)$. We shall divide the proof in two cases.

First case : F is incompressible in N. Then the action of $\pi_1(F)$ on $\mathcal{T}_F$ is small. By Skora's theorem [Sk], there are a measured geodesic lamination β on F and an isomorphism $\phi : \mathcal{T}_\beta \to \mathcal{T}_F$ from the dual tree $\mathcal{T}_\beta$ of β to $\mathcal{T}_F$. If β intersects α transversely, then α is realised in $\mathcal{T}$ (cf. [Ot3]). If β and α are disjoint, then α is collapsed in $\mathcal{T}$. It remains to deal with the case when α is a connected component of β. Let (η_i) be a sequence of positive numbers converging to 0. Let $\kappa_i \subset F$ be a sequence of segments intersecting α transversely such that $\kappa_{i+1} \subset \kappa_i$ and $\int_{\kappa_i} d\beta \leq \frac{1}{2}\eta_i$. Let θ_i be a train track minimally carrying α and having only one switch which is κ_i (refer to

[BoO, § 3.2] for the construction of such a train track). Let p' be a point of $\bigcap_i \kappa_i$, and $\hat{p}' \in \mathbb{H}^2$ a lift of p'. This point $\hat{p}'$ corresponds to a point of $\mathcal{T}_\beta$ which we shall also denote by $\hat{p}'$. Let $p \in \mathcal{T}$ be the image of $\hat{p}'$ under ϕ. Let $\hat{\kappa}_i \subset \mathbb{H}^2$ be the lift of κ_i that contains $\hat{p}'$, and $\hat{\theta}_i$ the lift of θ_i that contains $\hat{\kappa}_i$. Define ϕ_i on $\hat{\kappa}_i$ by $\phi_i(\hat{\kappa}_i) = p$. Extend ϕ_i to an equivariant map from the union of switches of $\hat{\theta}_i$ to $\mathcal{T}$. If $\hat{b}$ is a branch of $\hat{\theta}_i$, we define $\phi_i(\hat{b})$ to be the segment of $\mathcal{T}$ which connects the images of the vertical sides of $\hat{b}$. Finally, extend ϕ_i to a $\pi_1(S)$-equivariant map $\phi_i : \mathbb{H}^2 \to \mathcal{T}$. Let $\hat{b}$ be a branch of $\hat{\theta}_i$. Translating it by an element of $\pi_1(S)$, we can assume that $\hat{\kappa}_i$ contains a vertical side of $\hat{b}$. Since θ_i has only one switch, there is some $g \in \pi_1(S)$ such that $g(\hat{\kappa}_i)$ contains the other vertical side of $\hat{b}$. Let $\hat{k}_1$ be an arc joining $\hat{\kappa}_i$ to $g(\hat{\kappa}_i)$ whose projection k_1 on S lies in $b - |\beta|$. Then we have $\int_{k_1} d\beta = 0$. Let $\hat{k}_2 \subset \hat{\kappa}_i$ be an arc joining $\hat{p}'$ to $\hat{k}_1$ and let $\hat{k}_3 \subset g(\hat{\kappa}_i)$ be an arc joining $g(\hat{p}')$ to $\hat{k}_1$. The β-measures of k_2 and k_3 are less than $\int_{\kappa_i} d\beta \leq \frac{1}{2}\eta_i$. Therefore we have $\int_{k_1 \cup k_2 \cup k_3} d\beta \leq \eta_i$. This implies that the distance between $\hat{p}'$ and $g(\hat{p}')$ in $\mathcal{T}_\beta$ is less than η_i. It follows then from the construction of ϕ_i that the length of $\phi_i(\hat{b})$ is less than η_i. Thus we conclude that when F is incompressible, one of the situations of our lemma holds.

Second case : F is compressible in N. We shall use the assumption that λ lies in $\mathcal{D}(M)$ and some results of [KlS] to reduce this case to the previous case when F is incompressible. In the following, we shall be lead to consider the case when we can construct some homoclinic leaves using the results of [KlS]. Let us first consider what will happen if F contains a homoclinic leaf h.

Claim 3.2. *If F contains a homoclinic leaf h which does not intersect $|\alpha|$ transversely, then $S(\alpha)$ is incompressible, the closure of h contains two geodesic laminations $|\alpha|$ and A, and there is an essential I-bundle $S \times I$ in N over a surface S homeomorphic to $S(\alpha)$ such that $S \times I \cap \partial N = S \times \{0, 1\}$, $S \times \{0\} = S(\alpha)$, and A is arational in $S \times \{1\}$.*

Proof. As we saw in the section 2.6, an incompressible surface can not contain a homoclinic leaf. By construction, $F - S(\alpha)$ is incompressible, therefore the geodesic h does not lie in $F - \bar{S}(\alpha)$. Since α is arational in $S(\alpha)$, this implies that a half-leaf h_+ of h is asymptotic to a half-leaf of α in ∂N. Let h_- be a half-leaf of h which is disjoint from h_+.

Let us first assume that h_- is asymptotic to a half-leaf of α. By shortening h_- and h_+ if necessary, we can assume that they both lie in $S(\alpha)$ and are disjoint from λ. Since h is homoclinic, it follows then from Lemma 2.6 that h_- is asymptotic to h_+ in ∂N. In particular, since h_- is asymptotic to a half-leaf of α, the two half-leaves h_- and h_+ are asymptotic to the same half-leaf of α. Take a short arc k connecting h_- and h_+ so that they are lifted to a triangle with one vertex at infinity in $\partial\widetilde{N}$. Then any lift of a half-leaf of α going into the inside of this triangle must have the same endpoint as the lifts of h_-, h_+. This means that such a half-half leaf

captured between h_- and h_+ intersects a short arc k however short we take k to be. In other words $\int_k d\alpha$ is independent of our choice of k. Since α is not a closed curve and the length of k can be made to tend to 0, we have $\int_k d\alpha = 0$. Therefore, the union of k and the bounded component of $h - \partial k$ is homotopic to a meridian $m \subset F$ with $i(m, \alpha) = 0$. It follows that $m \subset F - S(\alpha)$, contradicting the fact that $F - S(\alpha)$ is incompressible.

If h_- is not asymptotic to a half-leaf of α, by shortening h_-, we can assume that h_- lies in $F - S(\alpha)$. If $S(\alpha)$ is compressible then by Lemma 2.8, $S(\alpha)$ contains a meridian m with respect to which α is in tight position. This implies that h_+ is also in tight position with respect to m. Since h is homoclinic, h_- also intersects m contradicting the assumption $h_- \subset F - S(\alpha)$.

If $S(\alpha)$ is incompressible, let $\mathcal{V} \subset F$ be a small annular neighbourhood of $\partial' \bar{S}(\alpha)$. By Proposition 2.2, $h_- \cup h_+$ lies in the boundary of a component of the characteristic submanifold of $(N, F - \mathcal{V})$. More precisely, there is an essential I-bundle in $(N, F - \mathcal{V})$ whose boundary contains $h_- \cup h_+$. Since h_- is not asymptotic to a half leaf of α, this I-bundle is not twisted. Hence it is homeomorphic to $S \times I$ with $S \times \{0\} = S(\alpha)$. Let $q : S \times I \to S$ be the projection along the fibres. Since h is homoclinic, the half-geodesics $q(h_+)$ and $q(h_-)$ are asymptotic. Furthermore, $q(h_+)$ is asymptotic to $q(\alpha)$. So $q(h_-)$ is asymptotic to $q(\alpha)$. It follows that $A = S \times \{1\} \cap q^{-1}(q(\alpha))$ is an arational lamination (in $S \times \{1\}$) that lies in the closure of h_-. This concludes the proof of Claim 3.2. □

Let (L_n) be a sequence of multi-curves in $S(\alpha)$ converging to the support of α in the Hausdorff topology. If N contains an incompressible I-bundle $S \times I$ with $S \times \{0\} = S(\alpha)$, then let $c \subset F - \bar{S}(\alpha)$ be a simple closed curve such that $c \subset S \times \{1\}$ and c cannot be homotoped into a component of $\partial S \times \{1\}$. Otherwise we set c to be empty. By [MoO], there exist a measured geodesic lamination $\beta_n \in \mathcal{ML}(F)$ and a morphism $\phi_n : \mathcal{T}_{\beta_n} \to \mathcal{T}_N$ from the dual tree of β_n to $\mathcal{T}_N$ such that for any closed curve l_n which lies in $L_n \cup \partial' \bar{S}(\alpha) \cup c$, either $\delta_{\mathcal{T}}(l_n) > 0$ and the restriction of ϕ_n to the axis of l_n is an isometry or $\delta_{\mathcal{T}}(l_n) = 0$ and $i(l_n, \beta_n) = 0$ (see [KlS] or [Le1]). We extract a subsequence so that $|\beta_n|$ converges in the Hausdorff topology to some geodesic lamination B.

If α does not intersect B, then α does not intersect β_n for sufficiently large n. This implies that α is collapsed in $\mathcal{T}_N$ and therefore in $\mathcal{T}$. If α intersects B transversely, then α intersects β_n transversely for sufficiently large n. In this case α is realised in $\mathcal{T}_N$ (cf. [KlS, Lemma 11]) and therefore α is realised in $\mathcal{T}$. It remains to deal with the case when $|\alpha|$ is contained in B.

For $n \in \mathbb{N}$, let β_n be the measured geodesic lamination which we have constructed above. Since we are considering the case when F is compressible, by Claim 2.3 either there is a meridian $m \subset F$ such that β_n contains no m-waves or there is a sequence of meridians converging in the Hausdorff

topology to a geodesic lamination which does not intersect β_n transversely. By the proof of [KlS, Proposition 2], if β_n intersects a meridian m and contains no m-waves, then $|\beta_n|$ can be extended to a geodesic lamination with a homoclinic leaf h_n. Furthermore by the proof of [KlS, Proposition 1], any neighbourhood of h_n contains a meridian. Thus we have found in any case a sequence of meridians converging in the Hausdorff topology to a geodesic lamination H which does not intersect B transversely. By Casson's criterion (Lemma 2.4), the lamination H contains a homoclinic leaf.

Recall that we are considering the case when $|\alpha| \subset B$. Therefore by Claim 3.2, $S(\alpha)$ is incompressible, H contains two geodesic laminations $|\alpha|$ and A and there is an essential I-bundle $S \times I$ over a surface S homeomorphic to $S(\alpha)$ such that $S \times \{0\} = S(\alpha)$ and A is arational in $S \times \{1\}$.

If $F - S(A)$ contains a meridian, then by Claim 2.3, either there is a meridian $m \subset F - S(A)$ such that α contains no m-waves or $F - S(A)$ contains a homoclinic leaf h which does not intersect α transversely.

Since $F - S(\alpha)$ is incompressible, in the former case m intersects α transversely. By construction the lifts of α and A to the universal cover of $S \times I$ have the same endpoints. Since α contains no m-waves and intersects m transversely, A also intersects m transversely (this is derived from the proof of [Le1, Affirmation 3.4]). This contradicts the fact that m lies in $F - S(A)$.

In the latter case, $F - S(A)$ contains a homoclinic leaf h which does not intersect α transversely. Applying Claim 3.2, we get an essential I-bundle $S \times I$ such that $S \times \{0\} = S(\alpha)$ and that $S \times \{1\} \subset F - S(A)$. By the uniqueness of the characteristic submanifold however (see [JaS] or [Jo]), this is impossible. Thus we have proved that $F - S(A)$ is an incompressible surface.

Any leaf of B intersecting $\partial(F - S(A))$ contains a half-leaf asymptotic to A. Such a half-leaf intersects infinitely many times the simple closed curve $c \subset (F - S(\alpha))$ (the one which we have chosen before constructing β_n). This implies that $i(\beta_n, \partial(F - S(A)))$ is $o(i(\beta_n, c))$. On the other hand, by construction, we have $i(\beta_n, c) = \delta_{\mathcal{T}}(c)$ for any n. Thus we get $i(\beta_n, \partial(F - S(A))) \longrightarrow 0$, which implies that the conjugacy class of $\pi_1(F)$ represented by each component of $\partial(F - S(A))$ has a fixed point in $\mathcal{T}$. This enables us to use Skora's Theorem on the minimal subtree of $\mathcal{T}$ which is invariant under the action of $\pi_1(F - S(A))$ and we can argue in the same way as when F is incompressible.

Thus we have proved the first part of Lemma 3.1. It only remains to show that at least one component of λ is realised in $\mathcal{T}$. This is proved in [Le2, Proposition 6.1] (see also [Le1, Proposition 6]). Let us briefly review the proof. Let (L_n) be a sequence of multi-curves converging to λ in the Hausdorff topology. As we have already seen, there exist a measured geodesic lamination $\beta_n \in \mathcal{ML}(\partial M)$ and a morphism $\phi_n : \mathcal{T}_{\beta_n} \to \mathcal{T}$ from the dual tree of β to $\mathcal{T}$ such that for any closed curve l_n which lies in L_n, either $\delta_{\mathcal{T}}(l_n) > 0$ and the restriction of ϕ_n to the axis of l_n is an isometry or

$\delta_{\mathcal{T}}(l_n) = 0$ and $i(l_n, \beta_n) = 0$. Extract a subsequence such that (β_n) converges in the Hausdorff topology to a geodesic lamination B. As we have seen above, any connected component of λ that intersects B transversely is realised in $\mathcal{T}$.

If $S(B)$ is compressible, then, by the proof of [KlS, Proposition 2], $S(B)$ contains a homoclinic leaf h which does not intersect B transversely. Such a homoclinic leaf intersects λ transversely ([Le2], see also the proof of Claim 3.2 above). Thus, if $S(B)$ is compressible, λ intersects B transversely.

If $S(B)$ is incompressible, then we can apply Skora's theorem to each component of $S(B)$. It follows that β_n does not depend on n for sufficiently large n. Denote by β this constant geodesic measured lamination β_n. We deduce then from [MoS2] that β is an annular lamination (see [BoO, Lemma 14] or [Le1, Lemme 4.4]). Thus we see that λ intersects transversely the support B of β also in this case. □

4. Mapping the train tracks to $\epsilon_n^{-1}\mathbb{H}^3$

In this section, we begin the proof Theorem 2. The first step in this proof is to construct a ρ_n-equivariant map from the universal cover of the train tracks constructed in Lemma 3.1 to $\mathbb{H}^3$ which has some nice properties. Recall that when $\hat{\tau}$ is a universal cover of a train track τ on a component S of ∂M, we say that a map $\hat{h} : \hat{\tau} \to \mathbb{H}^3$ is ρ_n-equivariant if and only if for every $g \in \pi_1(S)$ and $x \in \hat{\tau}$, we have $\hat{h}(gx) = \rho_n(i_*(g))\hat{h}(x)$, where i denotes the inclusion from S to M.

Now let us recall the statement of Theorem 2.

Theorem 2. *Let $\rho_n : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ be a sequence of discrete faithful representations which uniformise M and let $(\lambda_n) \subset \mathcal{ML}(\partial M)$ be a sequence of measured geodesic laminations such that $(l_{\rho_n}(\lambda_n))$ is a bounded sequence and (λ_n) converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination $\lambda \in \mathcal{D}(M)$. Then the sequence (ρ_n) has a compact closure in $AH(M)$, namely any subsequence contains an algebraically converging subsequence.*

Proof. Assume, seeking a contradiction, that no subsequences of (ρ_n) converge algebraically. Let us choose a non-principal ultrafilter ω. In the section 2 : Preliminaries, we have seen that (ρ_n), regarded as a sequence of actions on $\epsilon_n\mathbb{H}^3$, tends (with respect to ω) to a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$.

By Thurston's result (cf. [FLP] and [Pe]), weighted simple closed curves are dense in $\mathcal{ML}(S)$ for every component S of ∂M. By approximating each (λ_n) by a sequence of such unions of weighted simple closed curves, and by a diagonal extraction, we get a sequence of unions of weighted simple closed curves on components of ∂M satisfying the hypothesis of Theorem 2. Therefore we can assume that each $\lambda_n \cap S$ is a weighted simple closed curve if it is not empty for every component S of ∂M. We further extract a subsequence in such a way that $(|\lambda_n|)$ converges in the Hausdorff topology to a geodesic lamination L_∞. Let L_{rec} be the union of the recurrent leaves of

L_∞. We have $|\lambda| \subset L_{rec}$. Note that L_{rec} admits a transverse measure with full support whose restriction to $|\lambda|$ coincides with the transverse measure of λ, and endowed with this measure, L_{rec} is contained in $\mathcal{D}(M)$ since λ is contained in $\mathcal{D}(M)$.

Lemma 4.1. *After taking a subsequence of (ρ_n), there are train tracks τ_n, τ^1 and τ_n^2 such that*

- *each minimal sublamination of L_∞ is carried by τ^1 or τ_n^2;*
- *λ_n is minimally carried by τ_n;*
- *τ^1 and τ_n^2 are disjoint subtracks of τ_n;*
- *the switches of τ_n lie in $\tau^1 \cup \tau_n^2$;*
- *the sum of the weights with which τ_n carries λ_n is bounded independently of n;*

and there is a ρ_n-equivariant map $\tilde{h}_n : \hat{\tau}_n \to \mathbb{H}^3$ from the preimage of τ_n in the universal cover of ∂M to $\mathbb{H}^3$ such that :

a) *for any branch $\hat{b}$ of $\hat{\tau}_n$, its image $\tilde{h}_n(\hat{b})$ is either a geodesic segment or a point;*
b) *there are $R > 0$ and $n(R) \in \mathbb{N}$ such that, for $n \geq n(R)$, if $\hat{b}$ is a branch of $\hat{\tau}_n$ which projects into τ^1, then $l(\tilde{h}_n(\hat{b})) \geq R\epsilon_n^{-1}$;*
c) *there is a sequence of positive numbers $\delta_n \longrightarrow 0$ such that, for any $n \in \mathbb{N}$, if $\hat{b}_1, \hat{b}_2$ are two adjacent branches of $\hat{\tau}_n$ which are separated by a switch and which both project into τ^1, then the exterior angle between $\tilde{h}_n(\hat{b}_1)$ and $\tilde{h}_n(\hat{b}_2)$ is less than δ_n;*
d) *there is a sequence of positive numbers $\eta_n \longrightarrow 0$ such that, for any $n \in \mathbb{N}$, if $\hat{b}$ is a branch of $\hat{\tau}_n$ which projects into τ_n^2, then we have $\epsilon_n l(\tilde{h}_n(b)) \leq \eta_n$, where ϵ_n is the rescaling factor of $\mathbb{H}^3$ which appeared above;*
e) *for any $n \in \mathbb{N}$, if $\hat{b}$ is a branch of $\hat{\tau}_n$ which projects into $\tau_n - (\tau^1 \cup \tau_n^2)$, then $\lambda_n(b)(\epsilon_n l(\tilde{h}_n(\hat{b})))$ is less than η_n for (η_n) given in (d).*

Proof. Since we have only to construct train tracks on each component of ∂M with non-empty intersection with L_∞, we can assume that L_∞ is contained in a component S of ∂M. Let L be a minimal sublamination of L_{rec}.

Let us first consider the case when there is a train track θ minimally carrying L which is realised in $\mathcal{T}_S$ (recall that $\mathcal{T}_S$ is the minimal subtree of $\mathcal{T}$ invariant under $i_*\pi_1(S)$). Let $\hat{\theta}$ be a lift of θ to the universal cover $\mathbb{H}^2$ of S. There is a continuous $\pi_1(S)$-equivariant map $\phi_n : \mathbb{H}^2 \to \mathcal{T}_S$ such that ϕ_n is constant on every tie of $\hat{\theta}$ and the restriction of ϕ_n to any train route is injective. Following [Ot2], we shall construct a ρ_n-equivariant map $\tilde{h}_n : \hat{\theta} \to \mathbb{H}^3$. Let $\kappa_1, ..., \kappa_p$ be the switches of θ and $\hat{\kappa}_1, ..., \hat{\kappa}_p \subset \mathbb{H}^2$ lifts of $\kappa_1, ..., \kappa_p$. Denote by $(\tilde{x}_{i,n}) \in \mathcal{T}_S \subset (X_\omega, x)$ the point $\phi_n(\hat{\kappa}_i)$. We first define $\tilde{h}_n$ on $\{\hat{\kappa}_1, ..., \hat{\kappa}_p\}$ by setting $\tilde{h}_n(\hat{\kappa}_i) = \tilde{x}_{i,n}$. We extend this map to $\pi_1(S)(\{\hat{\kappa}_1, ..., \hat{\kappa}_p\})$ by $\tilde{h}_n(g(\kappa_i)) = \rho_n(g) \circ \tilde{h}_n(\hat{\kappa}_i)$ for any $g \in \pi_1(S)$ and

any $1 \leq i \leq p$. Let $\hat{b}$ be a branch of $\hat{\theta}$. The vertical sides of $\hat{b}$ lie in two switches $\hat{\kappa}$ and $\hat{\kappa}'$ whose images by $\tilde{h}_n$ have already been defined. On $\hat{b}$, we let $\tilde{h}_n$ be the map which is constant on each tie of $\hat{b}$, and which induces a parametrisation of the geodesic segment joining $\tilde{h}_n(\hat{\kappa})$ to $\tilde{h}_n(\hat{\kappa}')$ with constant speed on a horizontal side of $\hat{b}$. Then for any branch $\hat{b}$ of $\hat{\theta}$, we have $\epsilon_n l(\tilde{h}_n(\hat{b})) \longrightarrow l_{\mathcal{T}_S}(\phi_n(\hat{b})) > 0$.

Let θ' be the first subdivision of θ as defined in [Ot3, Chapitre 4, §4.1], and $\hat{\theta}' \subset \hat{\theta}$ a lift of θ'. We shall deform the map $\tilde{h}_n$ to one which is adapted to $\hat{\theta}'$. For a branch $\hat{b}$ of $\hat{\theta}'$, its image by $\tilde{h}_n$ is a broken geodesic segment, more precisely, it is the union of two geodesic segments. We deform $\tilde{h}_n$ by a homotopy which is constant on the vertical sides of $\hat{b}$ to a map which is constant on each tie of $\hat{b}$ and takes $\hat{b}$ to the geodesic segment joining the images under $\tilde{h}_n$ of the vertical sides of $\hat{b}$.

Since θ is realised in $\mathcal{T}_S$, it follows from the argument of [Ot3, Chapitre 4] that $\tilde{h}_n$ has the following properties :

b) there are $R > 0$ and $n(R)$ such that, for $n \geq n(R)$, if $\hat{b}$ is a branch of $\hat{\theta}'$, we have $l(\tilde{h}_n(\hat{b})) \geq R\epsilon_n^{-1}$;
c) there is a sequence of positive numbers $\delta_n \longrightarrow 0$ such that if $\hat{b}_1, \hat{b}_2$ are two adjacent branches of $\hat{\theta}$ which are separated by a switch, then the external angle between $\tilde{h}_n(\hat{b}_1)$ and $\tilde{h}_n(\hat{b}_2)$ is smaller than δ_n.

We perform the same for all the components of L_{rec} that are realised in $\mathcal{T}_S$. Denote by τ^1 the union of the train tracks θ' thus obtained and by $\tilde{h}_n : \hat{\tau}^1 \to \mathbb{H}^3$ the maps which agree with the map defined above on each connected component of $\hat{\tau}^1$. By Lemma 3.1, τ^1 is not empty. We also see that λ_i passes through every branch of τ^1 for every i after taking a subsequence.

When a component L of L_{rec} is not realised in $\mathcal{T}$, Lemma 3.1 gives rise to a sequence of train tracks θ_i each carrying L minimally. We can assume that λ_i passes through every branch of θ_i. Let us denote by τ_i^2 the union of the train tracks thus obtained from the components of L_{rec} which are not realised in $\mathcal{T}$. Finally we add branches to $\tau^1 \cup \tau_i^2$ to get a train track τ_i which minimally carries L_∞ and λ_i.

We shall now extend the map $\tilde{h}_n$ to the preimage of τ_i^2. Consider a connected component L of L_{rec} which is not realised in $\mathcal{T}$. Consider the subtrack θ_i of τ_i which minimally carries L. We get from Lemma 3.1 that there are a point $x \in \mathcal{T}$, a sequence $\eta_i \to 0$, and a sequence of $\pi_1(S)$-equivariant maps $\phi_i : \mathbb{H}^2 \to \mathcal{T}$ such that ϕ_i maps each branch of the preimage of θ_i to a geodesic segment (which may be a point) with length smaller than η_i and a lift of κ_i is mapped to x under ϕ_i. Set $x = (\tilde{x}_n) \in \Pi_n(\epsilon_n \mathbb{H}^3)$ and fix some $i \in \mathbb{N}$. Let $\hat{\theta}_i \subset \mathbb{H}^2$ be a lift of θ_i, and $\hat{\kappa}_i \subset \hat{\theta}_i$ the lift of κ_i that is mapped to x by ϕ_i. Let $G_i \subset \pi_1(\partial M)$ be a finite set consisting of all $g \in \pi_1(\partial M)$ such that if $\hat{b}$ is a branch of $\hat{\theta}_i$ and $\hat{\kappa}_i$ contains a vertical side of $\hat{b}$, then the other vertical side of $\hat{b}$ lies in $g(\hat{\kappa}_i)$. Recall that for each branch

$\hat{b}$, the length of $\phi_i(\hat{b})$ is less than η_i. Therefore, we have $d(x, gx) \leq \eta_i$ for any $g \in G_i$. Since $\mathcal{T}$ is the ω-ultralimit of $\epsilon_n \mathbb{H}^3$, we have $\epsilon_n d(\tilde{x}_n, \rho_n(g)(\tilde{x}_n)) \leq 2\eta_i$ for any $g \in G_i$ for n large enough. For any $g \in G_i \cup \{id\}$, we define $\tilde{h}_n(g(\hat{\kappa}_i))$ by $\tilde{h}_n(g(\hat{\kappa}_i)) = \rho_n(g)(\tilde{x}_n)$. Let $\hat{b}$ be a branch of $\hat{\theta}_i$ with vertical sides lying in two switches $\hat{\kappa}_i$ and $g(\hat{\kappa}_i)$ for some $g \in G_i$. If $\tilde{h}_n(\hat{\kappa}_i) = \tilde{h}_n(\rho_n(g)(\hat{\kappa}_i))$, then we set $\tilde{h}_n(\hat{b}) = \tilde{h}_n(\hat{\kappa}_i)$. Otherwise, we set $\tilde{h}_n$ to be the map which is constant on each tie of $\hat{b}$ and takes $\hat{b}$ to the geodesic segment joining $\tilde{h}_n(\hat{\kappa}_i)$ to $\tilde{h}_n(\rho_n(g)(\hat{\kappa}_i))$. Extend $\tilde{h}_n$ to an equivariant map from $\hat{\theta}_i$ to $\mathbb{H}^3$. For sufficiently large n and any branch $\hat{b}$ of $\hat{\theta}_i$, we have $\epsilon_n l(\tilde{h}_n(b)) \leq 2\eta_i$. Furthermore the sum of the weights with which θ_i carries λ_n is less than $\int_{\kappa_i} d\lambda_n \leq \int_{\kappa_1} d\lambda_n \longrightarrow \int_{\kappa_1} d\lambda$.

We do the same construction for all the components of L_{rec} that are not realised in $\mathcal{T}_S$, and denote by $\tilde{h}_n : \hat{\tau}^1 \cup \hat{\tau}_i^2 \to \mathbb{H}^3$ the maps whose restriction to each connected component of $\hat{\tau}^1 \cup \hat{\tau}_2$ is the maps thus defined. It follows from the construction that there is $N(i)$ such that for $n \geq N(i)$, if $\tilde{b}$ is a lift of a branch of τ_i^2, we have $\epsilon_n l(\tilde{h}_n(\tilde{b})) \leq 2\eta_i$.

It remains to define $\tilde{h}_n$ on the lifts of the branches of $\tau_i - (\tau^1 \cup \tau_i^2)$. Let $\hat{b}$ be such a lift. Let $\hat{\kappa}$ and $\hat{\kappa}'$ be the two vertical sides of $\hat{b}$. Their projections κ and κ' lie in $\tau^1 \cup \tau_i^2$. Hence their images by $\tilde{h}_n$ are already defined, and there are two points $x = (\tilde{x}_n), x' = (\tilde{x}'_n)$ in $\mathcal{T}$ such that $\tilde{h}_n(\bar{\kappa}) = \tilde{x}_n$ and $\tilde{h}_n(\bar{\kappa}') = \tilde{x}'_n$. We set $\tilde{h}_n$ to be the map which is constant on each tie of $\hat{b}$ and takes $\hat{b}$ to the geodesic segment joining $\tilde{x}_n$ to $\tilde{x}'_n$. We then have $\epsilon_n d(\tilde{x}_n, \tilde{x}'_n) \longrightarrow d(x, x')$. Furthermore, since λ is carried by $\tau^1 \cup \tau_i^2$, we have $\lambda_n(b) \longrightarrow 0$. So, for n large enough, we have $\lambda_n(b)(\epsilon_n l(\tilde{h}_n(\tilde{b}))) \leq 2\eta_i$.

Thus we have proved that there is $N(i)$ such that for $n \geq N(i)$, for a branch b of $\tau_i - \tau^1$, either b is a branch of τ_i^2 and we have $\epsilon_n l(\tilde{h}_n(\hat{b})) \leq 2\eta_i$ or b is a branch of $\tau_i - (\tau^1 \cup \tau_i^2)$ and we have $\lambda_n(b)(\epsilon_n l(\tilde{h}_n(\hat{b}))) \leq 2\eta_i$. Now by choosing $N(i)$ such that $N(i) < N(i+1)$, and taking a subsequence $\lambda_{N(n)}$ so that the n-th term is the original $N(n)$-th term, we obtain the expected train track. □

Since the $\lambda_n \cap S$ are weighted simple closed curves, $\tau_n \cap S$ is connected. Assume that $\tau_n \cap S \subset \tau^1$ for some component S of ∂M. For large n, λ_n is carried by τ^1. By [Ot2, Chapter 4], there is $\xi > 0$ such that for n large enough $l_{\rho_n}(\lambda_n) \geq \xi l_{\rho_n}(h_n(\lambda_n))$. It then follows from the condition (b) that $l_{\rho_n}(h_n(\lambda_n))$ tends to ∞. Therefore we have $l_{\rho_n}(\lambda_n) \geq \xi l_{\rho_n}(h_n(\lambda_n)) \longrightarrow \infty$, contradicting the assumption that $(l_{\rho_n}(\lambda_n))$ is a bounded sequence. From now on, we continue our argument under the assumption that $\tau^1 \cap S \subsetneq \tau_n \cap S$ for every component S of ∂M.

5. Constructing annuli

In this section, we shall work under the following assumptions to which we shall refer as the assumptions of the section 5:

Let $(\rho_n : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3))$ be a sequence of discrete faithful representations that uniformise M such that the action of $\rho_n(\pi_1(M))$ on $\epsilon_n \mathbb{H}^3$ tends (with respect to ω) to a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$. Let $\lambda_n \subset \mathcal{ML}(\partial M)$ be a sequence of measured geodesic laminations such that $l_{\rho_n}(\lambda_n)$ is a bounded sequence and that λ_n converges in $\mathcal{ML}(\partial M)$ to some measured geodesic lamination λ.

At the end of this section, we shall reach a contradiction by showing that, under these assumptions, λ cannot lie in $\mathcal{D}(M)$. This will conclude the proof of Theorem 2.

Take a component S of ∂M such that $S \cap \tau^1 \neq \emptyset$. In this section, we have only to pay attention to the behaviour of λ_n on S. Therefore, for simplicity, we denote $\lambda_n \cap S$ by λ_n, and $\tau_n \cap S$ by τ_n, etc. In particular, λ_n is assumed to be a weighted simple closed curve.

Let us denote by $c_n \subset S$ the support of λ_n and by w_n the weight of λ_n on c_n. Let c_n^* be the closed geodesic representative of c_n in $\mathbb{H}^3/\rho_n(\pi_1(M))$ and let
$h_n : \tau_n \to \mathbb{H}^3/\rho_n(\pi_1(M))$ be the projection of $\tilde{h}_n$. By construction, τ_n carries c_n. We set $c_n^1 = c_n \cap \tau^1$, $c_n^2 = c_n \cap \tau_n^2$ and $c_n^3 = c_n \cap (\tau_n - (\tau^1 \cup \tau_n^2))$. Then we have $w_n l_{\rho_n}(h_n(c_n^2)) = \sum_{b\text{ : the branches of }\tau_n^2} \lambda_n(b) l_{\rho_n}(h_n(b))$ and we have $w_n l_{\rho_n}(h_n(c_n^3)) = \sum_{b\text{: the branches of }\tau-(\tau_1\cup\tau_n^2)} \lambda_n(b) l_{\rho_n}(h_n(b))$. Therefore, by the property (d), we have $w_n l_{\rho_n}(h_n(c_n^2)) = o(\epsilon_n^{-1})$ and by the property (e), we have
$w_n l_{\rho_n}(h_n(c_n^3)) = o(\epsilon_n^{-1})$.

The curve $h_n(c_n)$ is a piecewise geodesic, whose geodesic segments we call edges and whose points where it failed to be a geodesic we call vertices. The vertices are the images of the intersection of c_n and the switches of τ_n. Let $x_{n,1}, ..., x_{n,p_n}$ be the vertices of $h_n(c_n)$, and choose p_n points $y_{n,1}, ..., y_{n,p_n}$ on c_n^*. For $1 \leq i \leq p_n$, we consider the geodesic triangle with vertices $y_{n,i}, x_{n,i}, x_{n,i+1}$ (with $x_{n,p_n+1} = x_{n,1}$ and $y_{n,p_n+1} = y_{n,1}$) and the geodesic triangle with vertices $x_{n,i+1}, y_{n,i}, y_{n,i+1}$. The union of these triangles for $i = 1, \ldots, p_n$ is a simplicial annulus $A_n = S^1 \times [0,1]$ cobounded by c_n^* and $h_n(c_n)$. The metric ν_n induced on this annulus by the lengths of paths is a hyperbolic metric with piecewise geodesic boundary. By the Gauss-Bonnet formula, the area of A_n is less than $2p_n\pi$. By Lemma 4.1, the sequence $(w_n p_n) = (\sum_{b\text{: the branches of }\tau_n} \lambda_n(b))$ is bounded. We parametrise A_n by $S^1 \times [0,1]$ so that the projection of $S^1 \times \{1\}$ to $\mathbb{H}^3/\rho_n(\pi_1(M))$ is c_n^*. We denote by $\bar{c}_n^j$ the part of $S^1 \times \{0\}$ corresponding to $h_n(c_n^j)$ for $j = 1, 2, 3$, and by $\bar{c}_n$ the union of the three.

For a positive number ϵ, which we shall specify later, and each point $x \in \bar{c}_n^1$, we consider a geodesic arc a_x on $(S^1 \times I, \nu_n)$ perpendicular to $S^1 \times \{0\}$ at x having length ϵ with respect to ν_n. If the perpendicular reaches $S^1 \times \partial I$ before the length ϵ is attained, we define a_x to be the geodesic arc having both endpoints on $S^1 \times \partial I$.

We are going to estimate from below the length of the set of points x for which the a_x reach $S^1 \times \{1\}$ without intersecting themselves or each other. Since the length of this set of points is bounded above by length(c_n^*) (with respect to ν_n), we get an inequality, which will appear as the inequality (i) below. For that, we need to subtract from the length of $\bar{c}_n^1$ the lengths of (I) the set of points x for which a_x has self-intersection, (II) the set of points x for which a_x intersects a_y with $x \neq y$, (III) the set of points x for which a_x has an endpoint on either $\bar{c}_n^2$ or $\bar{c}_n^3$, and (IV) the set of points x which are neither of type (I) nor of type (II) and for which a_x has an endpoint in the interior of $S^1 \times I$.

We first consider the contribution of the points of type (I) to the length, i.e., x for which a_x intersects itself transversely. By the Gauss-Bonnet formula, a geodesic loop formed by a subarc of a_x cannot be null-homotopic. Hence, there must be a loop formed by a subarc of a_x freely homotopic to $S^1 \times \{1\}$. It follows that if both of two perpendiculars a_{x_1}, a_{x_2} with $x_1 \neq x_2$ have self-intersection, then $a_{x_1} \cap a_{x_2} \neq \emptyset$. Thus, the contribution of the set of x with self-intersecting a_x to the length is absorbed in the contribution of x with a_x intersecting another a_y, that is, of type (II), which will be dealt with below.

We next consider the points x of type (II) namely a_x intersects a_y for some $y \in \bar{c}_n^1$. Let m be a point in the intersection $a_x \cap a_y$, and let a'_x, a'_y be subarcs of a_x, a_y between x and m and y and m respectively. Let β be an arc on $\bar{c}_n$ to which $a'_x \cup a'_y$ is homotopic fixing the endpoints. Suppose that β is also contained in $\bar{c}_n^1$. We then say that x is an *inessential* point of type (II). It was shown in [Bo, Lemme 5.11] that, in this situation, there is a constant ξ_n depending only on ϵ and the maximal exterior angle of the vertices on $\bar{c}_n^1$, which is less than δ_n in our case, such that x is within the distance ξ_n with respect to ν_n from a vertex of $\bar{c}_n^1$. It was also shown in [Bo] that the constant ξ_n tends to 0 as either δ_n or ϵ goes to 0. If β does not lie on $\bar{c}_n^1$, then we say that x is an *essential* point of type (II). We also call an arc such as $a'_x \cup a'_y$, which is not homotopic to an arc in $\bar{c}_n^1$, an *essential arc.* We should note that the length of such an essential arc is less than or equal to 2ϵ. Let $\bar{c}_n^+$ be the union of the essential point of type (II). We shall use the essential points of type (II) to construct a sequence of annuli in Lemma 5.2 and then these annuli will lead us to a contradiction.

Now we consider the points of type (III). The total length with respect to ν_n of the set of points x on $\bar{c}_n^1$ for which a_x reaches a point on $\bar{c}_n^2$ is bounded above by the length of $\bar{c}_n^2$. Similarly, the total length of the set of points x for which a_x reaches a point on $\bar{c}_n^3$ is bounded above by the length of $\bar{c}_n^3$.

Finally we consider the points of type (IV); the points x such that $a_x \setminus \{x\}$ is contained in the interior of $S^1 \times I$ while a_x has neither self-intersection nor intersection with another a_y. Since the union of a_x for x of type (IV) has area bounded below by the length of the set of points x of type (IV) multiplied by $\mathrm{sh}(\epsilon)$, we can bound the length from above by $\mathrm{Area}(A_n)/\mathrm{sh}(\epsilon)$.

Putting all of these considerations together, we get an inequality:

(i) $l_{\rho_n}(\bar{c}_n^1) - 2p_n\xi_n - l_{\rho_n}(\bar{c}_n^2) - l_{\rho_n}(\bar{c}_n^3) - \mathrm{Area}(A_n)/\mathrm{sh}(\epsilon) - l_{\rho_n}(\bar{c}_n^+) \leq l_{\rho_n}(c_n^*)$.

As we saw at the beginning of this section, from the properties (d) and (e), we get $w_n\epsilon_n l_{\rho_n}(\bar{c}_n^2) \longrightarrow 0$ and $w_n\epsilon_n l_{\rho_n}(\bar{c}_n^3) \longrightarrow 0$. We also saw that (w_np_n) is a bounded sequence. It follows that we have $2w_np_n\xi_n \longrightarrow 0$. This implies also that $w_n\mathrm{Area}(A_n) \leq 2w_np_n\pi$ is a bounded sequence and that we have $\epsilon_nw_n\mathrm{Area}(A_n) \longrightarrow 0$. By assumption $(w_nl_{\rho_n}(c_n^*)) = (l_n(\lambda_n))$ is a bounded sequence; hence $\epsilon_nw_nl_{\rho_n}(c_n^*)$ tends to 0. Thus we get $w_n\epsilon_n(l_{\nu_n}(\bar{c}_n^1) - l_{\nu_n}(\bar{c}_n^+)) \longrightarrow 0$. We put it into a form of a claim for later use.

Claim 5.1. *We have* $w_n\epsilon_n(l_{\nu_n}(\bar{c}_n^1) - l_{\nu_n}(\bar{c}_n^+)) \longrightarrow 0$.

From this claim, we shall deduce the following lemma which will prevent λ from lying in $\mathcal{D}(M)$.

Lemma 5.2. *Under the assumptions of the section 5, there is a homoclinic leaf or an annular lamination that does not intersect* $|\lambda|$ *transversely.*

Proof. We shall prove this lemma in several steps.

Fix an orientation for c_n. Let s be a segment lying in c_n^1, and put on it the orientation induced by that on c_n. The *train route* $b(1), ..., b(t)$ of s is the ordered finite sequence of branches of τ^1 through which s goes in this order : $b(i)$ is an element of the set $\{b_1, ..., b_p\}$ of branches of τ^1. We fix an orientation for each branch of τ^1. A branch $b(i)$ in the train route of s is said to be *positively oriented* if its orientation coincides with the orientation of s and *negatively oriented* otherwise. The *oriented train route* $bo(1), ..., bo(t)$ of s is the ordered finite sequence of oriented branches of τ^1 through which s goes in this order with the assigned orientations: $bo(i)$ is an element of $\{b_1, ..., b_p\} \times \{+, -\}$. When $(bo(i))_{i\in I}$ is an oriented train route, we shall denote by $(b(i))_{i\in I}$ the corresponding non-oriented train route.

Now we are going to construct some long and thin bands connecting two segments of $h_n(c_n)$.

Lemma 5.3. *There are two infinite oriented train routes* $bo, bo' : \mathbb{N} \to \{b_1, ..., b_p\} \times \{+, -\}$ *in* τ^1*, a map* $V : \mathbb{N} \to \mathbb{N}$*, and a map* $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that for any* $n, t \in \mathbb{N}$ *there are* $F(n, t)$ *disjoint segments* $s(n, j) \subset c_n^1$ $(1 \leq j \leq F(n, t))$ *satisfying the following :*

- *the oriented train route of* $s(n, j)$ *is* $(bo(i))_{i\leq t}$*;*
- *there is a segment* $s'(n, j) \subset c_n^1$ *whose oriented train route is* $(bo'(i))_{0\leq i\leq V(t)}$*;*
- *there is a homeomorphism* $g_{n,j} : h_n(s(n, j)) \to h_n(s'(n, j))$*;*
- $g_{n,j}(h_n(v(i))) \in h_n(b'(V(i)))$ *for any* $i \leq t$ *and* $n \in \mathbb{N}$ *where* $v(i) = b(i) \cap b(i + 1)$*;*
- *any point* $x \in h_n(s(n, j))$ *is connected to* $g_{n,j}(x)$ *by an essential arc* $\zeta_n(x)$ *on* A_n *with length less than* 6ϵ*;*
- *the simple closed curve* $s(n, j) \cup \zeta_n(\partial s(n, j)) \cup s'(n, j)$ *bounds a disc* $D_{n,j}$ *in* A_n*;*

- *for any $t \in \mathbb{N}$, $F(n,t)$ tends to ∞ as $n \longrightarrow \infty$.*

Notice that we have adopted the convention $0 \in \mathbb{N}$, which is necessary since the map F may vanish for many (n,t).

Proof. Let $\{\sigma_{1,n}, \sigma_{2,n}, \dots\} \subset \bar{c}_n$ be a maximal family of disjoint segments with diameter 6ϵ such that the middle point $x_{i,n}$ of $\sigma_{i,n}$ lies in $\bar{c}_n^+$. Consider a segment $\sigma_{i,n}$ and its middle point $x_{i,n}$. In the family of essential arcs joining $x_{i,n}$ to $\bar{c}_n^1$, we take an arc $a_{i,n}$ to be the shortest (with respect to ν_n). Since $x_{i,n}$ lies in $\bar{c}_n^+$, the length of $a_{i,n}$ is less than 2ϵ.

If $\partial a_{i,n} - x_{i,n}$ lies in some $\sigma_{j,n}$, we denote by $\zeta_{i,n}$ the geodesic segment joining $x_{i,n}$ to $x_{j,n}$ which is homotopic to $a_{i,n}$ relative to $x_{i,n} \cup \sigma_{j,n}$. If $\partial a_{i,n} - x_{i,n}$ is disjoint from $\bigcup \sigma_{j,n}$, we define $\zeta_{i,n}$ to be $a_{i,n}$. The length of each segment $\zeta_{i,n}$ thus obtained is less than 5ϵ. We shall next show that the segments $\zeta_{i,n}$ have mutually disjoint interiors.

Consider two different segments $\zeta_{1,n}$ and $\zeta_{2,n}$ and assume that their interiors intersect. Then the interiors of $a_{1,n}$ and $a_{2,n}$ also intersect. Let y be a point in the intersection. Let $[x_{i,n}, y[$ be the connected component of $a_{i,n} - \{y\}$ containing $x_{i,n}$. Let κ be the shortest segment among $a_{1,n} - [x_{1,n}, y[$ and $a_{2,n} - [x_{2,n}, y[$. Then, for $i = 1, 2$, the length of the arc $[x_{i,n}, y] \cup \kappa$ is less than or equal to the length of $a_{i,n}$. Furthermore one of the two arcs $[x_{i,n}, y] \cup \kappa$, say $[x_{1,n}, y] \cup \kappa$ is not the shortest in its homotopy class relative to the endpoints. Let a_1' be the shortest arc homotopic to $[x_{1,n}, y] \cup \kappa$ relative to the endpoints. Then the length of the segment a_1' (with respect to ν_n) is less than the length of $a_{1,n}$. Recall that we chose $a_{1,n}$ which is shortest among all essential arcs joining $x_{1,n}$ to $\bar{c}_n^1$. It follows that a_1' is not essential, namely there is a segment $\beta \subset \bar{c}_n^1$ homotopic to a_1' relative to the endpoints. The endpoints of β are $x_{1,n}$ and another point which we call y_1. The distance (with respect to ν_n) between $x_{1,n}$ and y_1 is less than the length of a_1' which is less than 2ϵ. By the properties (b) and (c), each component of $\bar{c}_n^1$ (in particular the one containing β) is a union of long geodesic segments such that the exterior angle between two consecutive segments is small. By [CEG, Lemma 4.2.10] such a component of $\bar{c}_n^1$ is a (K, η)-quasi-geodesic with $K \to 1, \eta \to 0$ as $n \to \infty$. It follows that there is N (independent of β) such that for $n \geq N$, the length of β is less than 3ϵ. This implies that y_1 lies in $\sigma_{1,n}$. By our definition of $\zeta_{1,n}$, it has an endpoint on $x_{1,n}$, not on y_1 (see figure 1). This contradicts our assumption that the interiors of $\zeta_{1,n}$ and $\zeta_{2,n}$ intersect.

Even if some segment $a_{i,n}$ has a self-intersection, the same argument shows that $\zeta_{i,n}$ does not have any self-intersection.

Thus we have proved the following claim.

Claim 5.4. *There is a map $f_n : \{x_{1,n}, x_{2,n}, \dots\} \to \bar{c}_n^1$ and a family (ζ_i) of essential segments with disjoint interiors such that the length of ζ_i is less than 5ϵ,*
$\partial \zeta_i = \{x_i, f_n(x_i)\}$, and $\{f_n(x_{1,n}), f_n(x_{2,n}), \dots\} \cap \bigcup_i \sigma_{i,n} \subset \{x_{1,n}, x_{2,n}, \dots\}$.

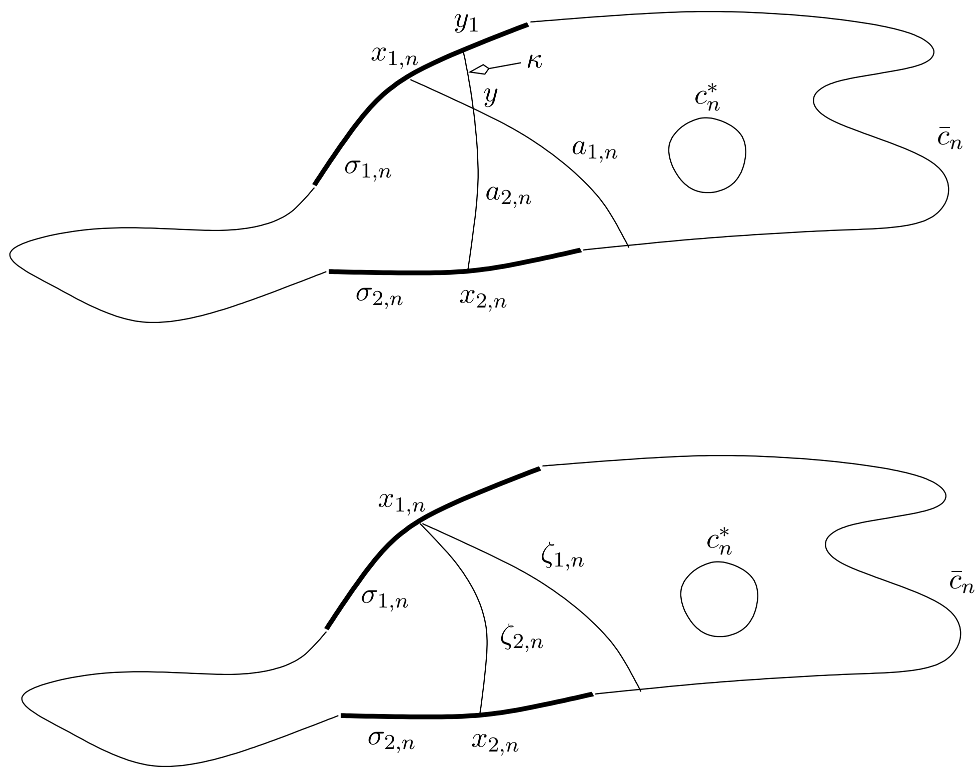

FIGURE 1. From $a_{i,n}$ to $\zeta_{i,n}$

Let j be an integer, and cut $\bar{c}_n^1$ into disjoint segments $s_{1,n}, s_{2,n}, \ldots$, each containing j edges (if the number of edges of some component of $\bar{c}_n^1$ is not a multiple of j, then there are some edges of $\bar{c}_n^1$ not belonging to any one of these segments). The following claim will help us to evaluate the number of segments thus obtained.

Claim 5.5. *Let r_n be the number of the components of $\bar{c}_n^1$. Then, $w_n r_n \longrightarrow 0$ as $n \to \infty$.*

Proof. Note that any train route on τ_n connecting a point in τ^1 and a point in τ_n^2 must pass through a point in $\tau_n - (\tau^1 \cup \tau_n^2))$. Therefore between any two components of c_n^1, there is a component of $c_n \cap (\tau_n - (\tau^1 \cup \tau_n^2))$. Hence $(w_n r_n)$ is bounded above by $\sum_{b \subset \tau_n - (\tau^1 \cup \tau_n^2)} \lambda_n(b)$. Since $|\lambda| \subset L_{rec}$ is carried by $\tau^1 \cup \tau_n^2$, the sum $\sum_{b \subset \tau_n - (\tau^1 \cup \tau_n^2)} \lambda_n(b)$ tends to 0 as $n \to \infty$. It follows that we have $w_n r_n \longrightarrow 0$. □

Since w_n(the number of edges of $\bar{c}_n^1$) $= w_n(\sum_{b \subset \tau^1} c_n(b)) \longrightarrow \sum_{b \subset \tau^1} \lambda(b)$, the number of edges of $\bar{c}_n^1$ is $\Theta(w_n^{-1}) = \Theta(p_n)$. The number of edges not lying in one of the $s_{i,n}$ is less than $j r_n = o(p_n)$. It follows that the number of segments $s_{i,n}$ is $\Theta(p_n)$.

Let t_n be the number of edges of $\bar{c}_n^1$ containing no segment among the $\sigma_{i,n}$. If an edge e contains no segment among the $\sigma_{i,n}$, then there is no point of $\bar{c}_n^+$ in e outside the 6ϵ-neighbourhood of ∂e. By the property (b), the total length of these edges is greater than $t_n R\epsilon_n^{-1}$ and smaller than $l_{\nu_n}(\bar{c}_n^1) - l_{\nu_n}(\bar{c}_n^+) + 12t_n\epsilon$. By Claim 5.1, we have $w_n\epsilon_n t_n R\epsilon_n^{-1} \longrightarrow 0$. Therefore t_n is $o(w_n^{-1}) = o(p_n)$.

Thus we have seen that among the $s_{i,n}$, there are $\Theta(p_n)$ disjoint segments lying in $\bar{c}_n^1$ and containing j edges each of which contains a segment among the $\sigma_{i,n}$. We shall denote these $\Theta(p_n)$-many segments again by $s_{1,n}, s_{2,n}, \dots$.

Let $s \in \{s_{1,n}, s_{2,n}, \dots\}$ be a segment with the following property : there are at least two distinct components of $\bar{c}_n^1$ containing an endpoint of $\zeta_{i,n}$ for some $x_{i,n} \in s$. Let t'_n be the number of those with this property among the $s_{i,n}$. In each such a segment s, we choose two points in $\{x_{1,n}, x_{2,n}, ...\} \cap s$, say $x_{1,n}, x_{2,n}$, such that $\zeta_{1,n}$ and $\zeta_{2,n}$ connect s to distinct components of $\bar{c}_n^1$ and the segment $]x_{1,n}, x_{2,n}[\subset s$ does not contain any $x_{k,n}$. There may be several $\zeta_{i,n}$ which have $x_{1,n}$ or $x_{2,n}$ as an endpoint. There are two points $y_{1,n}$ and $y_{2,n}$ which lie in two distinct components of $\bar{c}_n^1$ such that $y_{1,n}$ (resp. $y_{2,n}$) is connected to $x_{1,n}$ or $x_{2,n}$ by some $\zeta_{i,n}$ and that if $[y_{1,n}, y_{2,n}]$ is the segment of $\bar{c}_n$ joining $y_{1,n}$ to $y_{2,n}$ whose interior does not contain $x_{1,n}$, then there is no $\zeta_{i,n}$ connecting $]y_{1,n}, y_{2,n}[$ to $\{x_{1,n}, x_{2,n}\}$. Let us embed A_n in a round disc such that $\bar{c}_n$ is the boundary and connect $y_{1,n}$ to $y_{2,n}$ by a geodesic segment with respect to the ordinary Euclidean metric on the disc.

Assume that for another segment $s' \in \{s_{1,n}, \dots\}$ with the same property, the resulting geodesic segment intersects transversely the geodesic segment produced from s above (i.e. the geodesic segment connecting $y_{1,n}$ to $y_{2,n}$). Suppose that $x_{3,n}$ and $x_{4,n}$ are the points on s' chosen in the same way as $x_{1,n}, x_{2,n}$ for s. Then we have $[x_{3,n}, x_{4,n}] \subset [y_{1,n}, y_{2,n}]$ and either $y_{1,n} \in \{x_{3,n}, x_{4,n}\}$ or $y_{2,n} \in \{x_{3,n}, x_{4,n}\}$. There is only one segment $s_{i,n}$ containing $f(x_{1,n})$ and only one containing $f(x_{2,n})$. Since there is no ζ_i connecting $]y_{1,n}, y_{2,n}[$ to $x_{1,n}, x_{2,n}$, if we have $f(x_{3,n}) \in \{x_{1,n}, x_{2,n}\}$ then $x_{3,n}$ is equal to $f(x_{1,n})$ or $f(x_{2,n})$. It follows that there are only two possibilities for the configuration of s, s' and the geodesic segments constructed above (see Figure 2).

From that we deduce that there are at least $\frac{1}{3}t'_n$ disjoint geodesic segments in the round disc, each connecting two distinct components of $\bar{c}_n^1$. Furthermore, since the ζ_i have disjoint interiors, any pair of connected components of $\bar{c}_n^1$ is connected by at most two of these disjoint segments. Consider a map from $\bar{c}_n$ to a round circle that preserves the order and maps each connected component C_i of $\bar{c}_n^1$ to a point Q_i. Join two points Q_i and Q_j by a segment if and only if there is one of the segments constructed above which joins C_i and C_j. Thus we have constructed at least $\frac{1}{6}t'_n$ segments with disjoint interiors. We can now add some geodesic segments in the disc to get a triangulation of the polygon with vertices $Q_1, ..., Q_{r_n}$. Such a triangulation has $2r_n - 3$

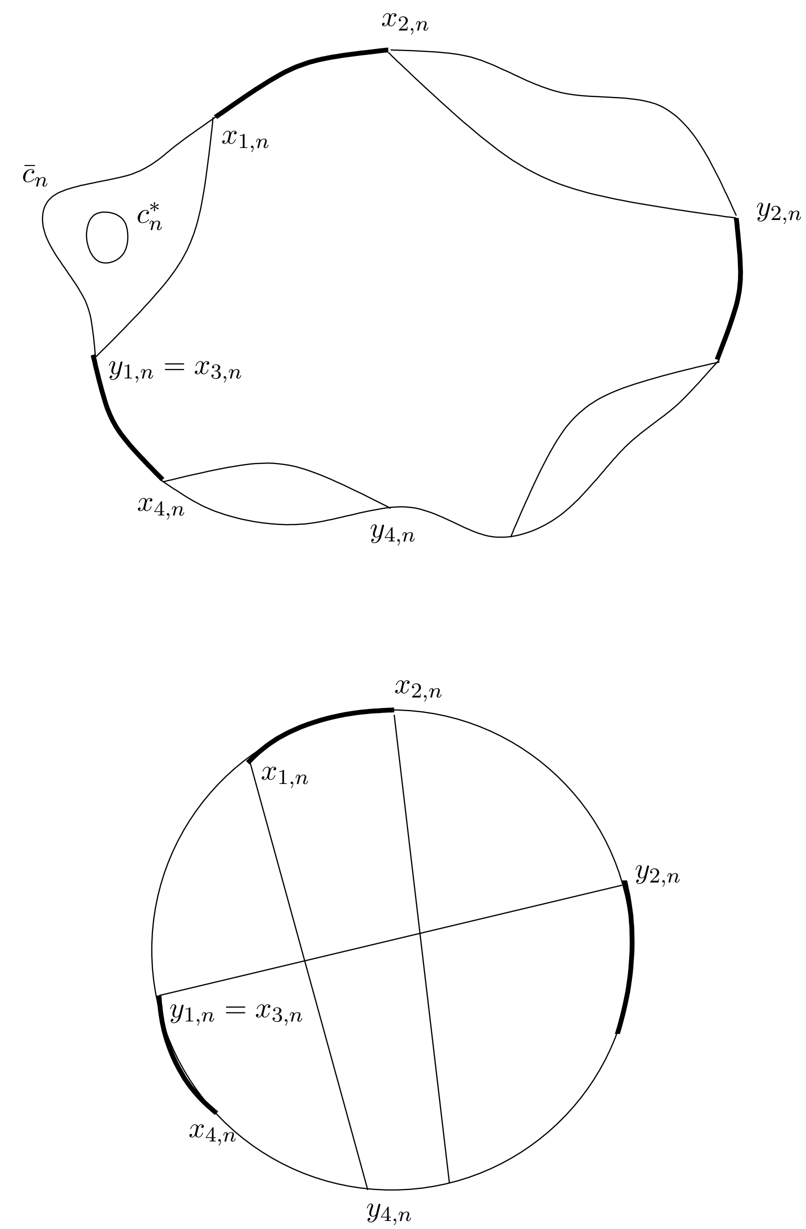

FIGURE 2. A segment intersects at most two other segments

edges (this can easily be computed with the Euler formula). Therefore we have $\frac{1}{6}t'_n \leq 2r_n - 3 = o(p_n)$.

Since we initially had $\Theta(p_n)$ segments $\{s_{1,n}, s_{2,n}, ...\}$, after excluding $o(p_n)$ segments as above from them, there remains $\Theta(p_n)$ disjoint segments $\{s_{1,n}, s_{2,n}, ...\}$ in $\bar{c}_n^1$, (for which we use the same symbols) such that if s is one of those segments, then we have :

- s contains j edges;
- i) each edge of s contains some $x_{i,n}$;
- ii) there is a unique component C (depending on s) of $\bar{c}_n^1$ such that for any $x_{i,n} \in s$, we have $f_n(x_{i,n}) \in C$.

Let s be one of these segments, C the associated component of $\bar{c}_n^1$, and x a point of $s \cap \{x_{1,n}, ...\}$. Denote by ζ_x the corresponding $\zeta_{i,n}$. Since A_n is an annulus and ζ_x is embedded, there are only two possibilities for the homotopy class of ζ_x relative to $s \cup C$. Therefore, taking $2j$ instead of j at the beginning and cutting each segment into halves, we get $\Theta(p_n)$ disjoint segments $\{s_{1,n}, ...\}$ in $\bar{c}_n^1$ each one containing j edges and satisfying (i), (ii) above and :

iii) for any $x, y \in s \cap \bar{c}_n^+$, ζ_x and ζ_y are homotopic relative to $s \cup C$.

Let s be one of the segments produced above, and C the corresponding component of $\bar{c}_n^1$. Let x and y be the extremal points of $s \cap \{x_{1,n}, ...\}$, and $[x, y]$ the segment in s joining x to y. The segment $[x, y]$ contains at least $(j-2)$ edges. We have the following :

Lemma 5.6. *There is $N \in \mathbb{N}$ which does not depend on s such that for any $n \geq N$ we have the following by homotoping A_n keeping ∂A_n and $\bar{c}_n^i$ unchanged:*

- there is a homeomorphism $g_n : [x, y] \to [f_n(x), f_n(y)]$ such that for any $n \geq N$ and any $z \in [x, y]$, the two points z and $g_n(z)$ are connected by an essential arc whose length (with respect to ν_n) is smaller than 6ϵ.

Proof. By the property (iii), the simple closed curve $\zeta_x \cup [x, y] \cup \zeta_y \cup [f_n(x), f_n(y)]$ bounds a disc in A_n. Since both $[x, y]$ and $[f_n(x), f_n(y)]$ lie in $\bar{c}_n^1$, by the properties (b) and (c), they consist of long geodesic segments such that the external angles formed by two adjacent segments are less than δ_n. Let k be the geodesic segment in $\mathbb{H}^3/\rho_n(\pi_1(M))$ joining x to y that is homotopic to $[x, y]$, and let k' be the one in $\mathbb{H}^3/\rho_n(\pi_1(M))$ joining $f_n(x)$ to $f_n(y)$ that is homotopic to $[f_n(x), f_n(y)]$. We parametrise the arcs $[x, y]$, $[f_n(x), f_n(y)]$, k and k' by the arc-length.

By [CEG, Lemma 4.2.10], for sufficiently large n, we have $d([x, y](t), k(t)) \leq \epsilon'_n$ and $d([f_n(x), f_n(y)](t), k'(t)) \leq \epsilon'_n$ for any t with $\epsilon'_n \longrightarrow 0$. It follows that $1 \leq \frac{l([x,y])}{l(k)} \leq \frac{l(k)+\epsilon'_n}{l(k)} \leq 1 + \epsilon'_n$ and that $1 \leq \frac{l([f_n(x), f_n(y))}{l(k')} \leq 1 + \epsilon'_n$ for sufficiently large n, where $l(.)$ denotes the length in $\mathbb{H}^3/\rho_n(\pi_1(M))$. Therefore, we have $d([x, y](\frac{l([x,y])}{l(k)} t), k(t)) \leq 2\epsilon'_n$ for sufficiently large n. For the same reason, we have also $d([f_n(x), f_n(y)](\frac{l([f_n(x), f_n(y)])}{l(k')} t), k'(t)) \leq 2\epsilon'_n$.

By the property (iii), the simple closed curve $\zeta_x \cup k \cup \zeta_y \cup k'$ bounds a disc. Since k and k' are geodesic segments, the function $d(k(t), k'(\frac{l(k')}{l(k)} t))$ is convex. Therefore we have $d(k(t), k'(\frac{l(k')}{l(k)} t)) \leq 5\epsilon$ for any t.

We define $g_n : [x, y] \to [f_n(x), f_n(y)]$ by the following formula $g_n([x, y](\frac{l([x,y])}{l(k)} t)) = [f_n(x), f_n(y)](\frac{l([f_n(x), f_n(y))}{l(k)} t)$. Setting $z = [x, y](\frac{l([x,y])}{l(k)} t)$,

we have that the distance $d(z, g_n(z))$ is less than the following quantity $d([x,y](\frac{l([x,y])}{l(k)}t), k(t)) + d(k(t), k'(\frac{l(k')}{l(k)}t)) + d(k'(\frac{l(k')}{l(k)}t, [f_n(x), f_n(y)](\frac{l([f_n(x),f_n(y)])}{l(k)}t))$. Then we get from the above $d(z, g_n(z)) \leq 5\epsilon + 4\epsilon'_n$. Now we conclude by taking N such that $4\epsilon'_n \leq \epsilon$ for $n \geq N$ and by changing A_n by a homotopy so that the geodesic segment $\zeta_n(x)$ connecting z to $g_n(z)$ lies in A_n for any z in s. □

We remove from s its two extremal edges so that g_n is defined on the entire s.

By construction, there is a constant R' such that for any branch $\hat{b}$ of $\hat{\tau}^1$, we have $l(\tilde{h}_n(\hat{b})) \leq R'\epsilon_n^{-1}$. By Lemma 5.6 and since $\delta_n \longrightarrow 0$ as $n \to \infty$, we have $l_{\rho_n}(g_n(s)) \leq l_{\rho_n}(s) + 13\epsilon \leq (j-2)R'\epsilon_n^{-1} + 13\epsilon$. Therefore if j'_n is the number of edges which $g_n(s)$ contains, $j'_n \leq (j-2)(\frac{R'}{R}+1)$ for large n.

Since τ^1 has only finitely many branches, there are only finitely many possibilities for the oriented train routes of s and of $g_n(s)$. Thus we can find $\Theta(p_n)$-many disjoint segments $\{s_{1,n}, ...\}$ with the properties (i), (ii) and (iii), having the same oriented train route such that the $g_n(s)$ also have the same oriented train route for all $s \in \{s_{1,n}, \dots\}$.

Now we fix j to be 3 and extract a subsequence so that the oriented train routes of $s_{i,n}$ and $g_n(s_{i,n})$ do not depend on n.

To each $s_{i,n}$, we add the edge of $\bar{c}_n$ which is adjacent to the last (with respect to the orientation of $s_{i,n}$) edge of $s_{i,n}$. In this family of segments, take a maximal family of disjoint segments lying in $\bar{c}_n^1$, which we denote by $\{s'_{1,n}, s'_{2,n}, ...\}$. By Claim 5.5, the number of elements in this family is $\Theta(p_n)$. We shall find in this family $\Theta(p_n)$ segments which have the same properties as the segments $\{s_{1,n}, \dots\}$ above.

We have already seen that the number of edges of $\bar{c}_n^1$ not containing a segment $\sigma_{i,n}$ is $o(p_n)$. Therefore we can find $\Theta(p_n)$ segments among the $s'_{i,n}$ each of whose edges contains an element of $\{\sigma_{1,n}, ...\}$. We denote this family of segments again by $s'_{i,n}$ changing the indices.

By the same argument as above (see the paragraph before the statement of property (ii)) the number of segments $s'_{i,n}$ such that there are two distinct components of $\bar{c}_n^1$ containing a point $f_n(x)$ for some $x \in s'_{i,n} \cap \{x_1, ...\}$ is $o(p_n)$. Therefore we have $\Theta(p_n)$ segments among the $s'_{i,n}$ satisfying (i) and (ii). We denote this subfamily of segments again by $s'_{i,n}$.

Let $s'_{1,n}$ be one of these segments, and C_1 the connected component of $\bar{c}_n^1$ such that we have $f_n(x) \in C_1$ for any $x \in s'_{1,n} \cap \{x_{1,n}, ...\}$. Assume that there are two arcs ζ_x and ζ_y with $x, y \in s'_{1,n} \cap \{x_{1,n}, ...\}$ which are not homotopic relative to $s'_1 \cup C_1$. Let $s'_{2,n}$ be another one of the $s'_{i,n}$ and C_2 the connected component of $\bar{c}_n^1$ such that $f_n(x) \in C_2$ for any $x \in s'_{2,n} \cap \{x_{1,n}, ...\}$. Since the $\zeta_{i,n}$ have disjoint interiors, if $C_1 = C_2$, then $s'_{2,n}$ has property (iii). It follows that there are at most $r_n = o(p_n)$ segments $s'_{i,n}$ that do not have property (iii).

Thus we have seen that there are $\Theta(p_n)$ disjoint segments $\{s'_{1,n}, ...\}$ with the properties (i), (ii) and (iii) such that each $s'_{i,n}$ is obtained by adding to one of the $s_{i,n}$ the edge adjacent to the last edge. The proof of Lemma 5.6 applies to these segments, yielding an homeomorphism g_n which can be chosen to coincide with the one defined on the segments $s_{i,n}$ if restricted to them.

From this last family, we take $\Theta(p_n)$ segments $s'_{i,n}$ such that the oriented train routes of $s'_{i,n}$ and $g_n(s'_{i,n})$ do not depend on i. Then we extract a subsequence (with respect to n) such that for sufficiently large n, the oriented train routes of $s'_{i,n}$ and of $g_n(s'_{i,n})$ do not depend on n.

By doing this argument recursively, increasing j one by one, we complete the proof of Lemma 5.3. $\square$

When $(bo(i))_{i\in\mathbb{N}}$ is an oriented train route, we denote by $(b(i))_{i\in\mathbb{N}}$ the corresponding non-oriented train route. Since for any t, the simple closed curve c_n goes through the oriented train routes $(bo(i))_{i\leq t}$ and $(bo'(i))_{i\leq t}$ for n large enough, there are two half-leaves l_+ and l'_+ of L_{rec} whose oriented train routes are $(bo(i))_{i\in\mathbb{N}}$ and $(bo'(i))_{i\in\mathbb{N}}$ respectively.

Let $e \in \mathbb{N}$ be an integer which we shall specify later, and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing map such that : (*)

(1) $bo(\varphi(i) + j) = bo(\varphi(0) + j)$ for any $i \in \mathbb{N}$ and any $0 \leq j \leq e$.
(2) $bo'(V(\varphi(i)) + j) = bo'(V(\varphi(0)) + j)$ for any $i \in \mathbb{N}$ and any $0 \leq j \leq \frac{R'}{R}e + 1$.
(3) Suppose that l_+ (resp. l'_+) is not a closed curve, and let α_i be a sub-arc of l_+ with train route $(b(j))_{\varphi(0)\leq j<\varphi(i)}$ (resp. α'_i the arc of l'_+ with train route $(b'(i))_{\varphi(0)\leq i<\varphi(i)}$). We can assume that the two endpoints of α_i (resp α'_i) lie in the same switch of τ^1. Then the sequences $(\partial\alpha_i)$ (resp. $\partial\alpha'_i$) converges to a single point with respect to the Hausdorff topology.

The existence of such a map φ follows from the fact that the number of branches of τ^1 is bounded.

For $t \in \mathbb{N}$, let $N(t)$ be an integer such that $F(n,t) \geq 1$ for any $n \geq N(t)$ where $F(.,.)$ is the map defined in Lemma 5.3. The map $N : \mathbb{N} \to \mathbb{N}$ can be assumed to be increasing and non-constant since we can assume that $F(n,t)$ is non-constant with respect to t. For $n \geq N(e)$, let k_n be the maximal number such that $N(\varphi(k_n) + e) \leq n < N(\varphi(k_n + 1) + e)$. Since $\tau^1 \subsetneq \tau$, the train route of c_n never contains the entire $(b(i))_{i\in\mathbb{N}}$, which lies in τ^1. This implies that $F(n,t) = 0$ for sufficiently large t; hence we have $k_n < \infty$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be the map defined by $\psi(n) = k_n$. Since φ is strictly increasing and N is increasing and non-constant, $\psi(n)$ tends to ∞ as n tends to ∞. To get a better idea on this map ψ, the reader should notice that if we forget the condition (3), $\psi(n)$ denotes the number of times that $s(n,j)$ comes back to the oriented train route $(bo(k))_{\varphi(0)\leq k\leq e+\varphi(0)}$ at the same times as $s'(n,j)$ comes back to the oriented train route $(bo'(k))_{\varphi(0)\leq k\leq \frac{R'}{R}e+1+\varphi(0)}$.

Let us denote by b' the branch $b'(V(\varphi(0)))$. For each $n \geq N(e)$, cut the geodesic segment $h_n(b')$ into $\psi(n)$ isometric segments $\varsigma(j,n)$. The length $l_{\rho_n}(\varsigma(j,n))$ of such a segment is $\frac{l_{\rho_n}(h_n(b'))}{\psi(n)} \leq R' \frac{\epsilon_n^{-1}}{\psi(n)} = o(\epsilon_n^{-1})$.

Let $s(n, \varphi(\psi(n)) + e) \subset c_n^1$ be a segment which we obtained in Lemma 5.3. For simplicity, we shall denote it by s_n and the corresponding segment $s'(n, \varphi(\psi(n)) + e)$ by s'_n. Note that s_n goes through the train route $(bo(k))_{0 \leq k \leq \varphi(\psi(n))+e}$, and hence it travels through $(bo(k))_{\varphi(0) \leq k \leq e+\varphi(0)}$ at least $\psi(n)+1$ times. The same thing holds for s'_n by the property (2) of (*). By Lemma 5.3, we see that for any $0 \leq i \leq \psi(n)$, $g_n \circ h_n \circ v(\varphi(i))$ lies in $h_n(b'(V(\varphi(i)))) = h_n(b')$ (where g_n stands for $g_{n,\varphi(\psi(n))+e}$). Recall that we have $\psi(n)$ segments $\varsigma(j,n)$. Among the $\psi(n)+1$ points $g_n \circ h_n \circ v(\varphi(i))$, $0 \leq i \leq \psi(n)$, we can find two points $g_n \circ h_n \circ v(\varphi(i_n))$ and $g_n \circ h_n \circ v(\varphi(j_n))$ which lie in the same segment among the $\varsigma(j,n)$, which we denote by ς_n. Let $I_n \subset s_n$ be the segment between $v(\varphi(i_n))$ and $v(\varphi(j_n))$. Assume that $j_n > i_n$ and let J_n be the sub-segment of s_n containing the e vertices following $v(\varphi(j_n))$.

Let $\tilde{s}_n \subset \mathbb{H}^3$ be a lift of $h_n(s_n)$, and let $\tilde{v}(\varphi(i_n))$, $\tilde{v}(\varphi(j_n))$, $\tilde{I}_n$ and $\tilde{J}_n$ be lifts of $h_n \circ v(\varphi(i_n))$, $h_n \circ v(\varphi(j_n))$, $h_n(I_n)$ and $h_n(J_n)$ respectively, lying in $\tilde{s}_n$. We lift the map g_n to a map $\tilde{g}_n$ from $\tilde{s}_n$ to a lift $\tilde{s}'_n$ of s'_n. Let $\rho_n(a_n) \in \rho_n(\pi_1(M))$ be the covering translation that takes $\tilde{v}(\varphi(i_n))$ to $\tilde{v}(\varphi(j_n))$. Since all the branches $bo(\varphi(i))$ have the same orientation, $\rho_n(a_n)$ acts on $\tilde{I}_n$ as a translation. Since $bo(\varphi(i_n)+j) = bo(\varphi(j_n)+j)$ for all $j \leq e$, the isometry $\rho_n(a_n)$ also acts as a translation on $\tilde{I}_n \cup \tilde{J}_n$. Let $\tilde{\varsigma}_n$ be the lift of ς_n that contains $\tilde{g}_n \circ \tilde{v}(\varphi(i_n))$, let $\tilde{\varsigma}'_n$ be the lift that contains $\tilde{g}_n \circ \tilde{v}(\varphi(j_n))$, and let $\rho_n(a'_n)$ be the isometry that maps $\tilde{\varsigma}_n$ to $\tilde{\varsigma}'_n$. Let $\tilde{I}'_n \subset \tilde{s}'_n$ be the piecewise geodesic segment between $\tilde{g}_n \circ \tilde{v}(\varphi(i_n))$ and $\rho_n(a'_n) \circ \tilde{g}_n \circ \tilde{v}(\varphi(i_n))$, and let $\tilde{J}'_n \subset \tilde{s}'_n$ be the segment between $\rho_n(a'_n) \circ \tilde{g}_n \circ \tilde{v}(\varphi(i_n))$ and $\tilde{g}_n \circ \tilde{v}(\varphi(j_n)+e)$. From the assumption that $bo'(V(\varphi(i_n))+j) = bo'(V(\varphi(j_n))+j)$ for any $0 \leq j \leq \frac{R'}{R}e+1$, it follows that $\rho_n(a'_n)$ acts as a translation on $\tilde{I}'_n \cup \tilde{J}'_n$.

As we saw before, the length of $\tilde{\varsigma}_n$ is $o(\epsilon_n^{-1})$. It follows that the distance $d(\tilde{g}_n \circ \tilde{v}(\varphi(j_n)), \rho_n(a'_n) \circ \tilde{g}_n \circ \tilde{v}(\varphi(i_n))$ is $o(\epsilon_n^{-1})$. From this and the facts that $\rho_n(a_n)$ acts as a translation on $\tilde{I}_n \cup \tilde{J}_n$ and that $\rho_n(a'_n)$ acts as a translation on $\tilde{I}'_n \cup \tilde{J}'_n$, we shall deduce the following claim :

Claim 5.7. *For $R > 0$, let $\mathcal{V}_R(\tilde{J}_n)$ be the R-neighbourhood of $\tilde{J}_n$, then for any sequence of points $\tilde{z}_n \in \mathcal{V}_R(\tilde{J}_n)$, we have $d(\tilde{z}_n, \rho_n(a_n^{-1}a'_n)(\tilde{z}_n))$ is $o(\epsilon_n^{-1})$.*

Proof. It is sufficient to prove this claim for any sequence $\tilde{z}_n \in \tilde{J}_n$. Since $\rho_n(a_n^{-1})$ acts as a translation on $\tilde{I}_n \cup \tilde{J}_n$, the point $\rho_n(a_n^{-1})(\tilde{z}_n)$ is the point $\tilde{z}'_n$ on $\tilde{I}_n \cup \tilde{J}_n$ that is at a distance (measured on $\tilde{I}_n \cup \tilde{J}_n$) equal to $d(\tilde{z}_n, \tilde{v}(\varphi(j_n))$ from $\tilde{v}(\varphi(i_n))$. Hence we have $d(\tilde{z}_n, \tilde{v}(\varphi(j_n)) = d(\tilde{z}'_n, \tilde{v}(\varphi(i_n)))$.

The point $\tilde{z}''_n = \rho_n(a'_n) \circ \tilde{g}_n(\tilde{z}'_n)$ is the point of $\tilde{J}'_n$ that is at a distance (measured on $\tilde{I}'_n \cup \tilde{J}'_n$) of $d(\tilde{g}_n(z'_n), \tilde{g}_n \circ \tilde{v}(\varphi(i_n))$ from $\rho_n(a'_n) \circ \tilde{g}_n \circ \tilde{v}(\varphi(i_n))$.

The points $g_n \circ h_n \circ v(\varphi(i_n))$ and $g_n \circ h_n \circ v(\varphi(j_n))$ both lie in ς_n whose length is $o(\epsilon_n^{-1})$. It follows that the distance $d(\tilde{g}_n \circ \tilde{v}(\varphi(j_n)), \rho_n(a'_n) \circ \tilde{g}_n \circ \tilde{v}(\varphi(i_n)))$ is $o(\epsilon_n^{-1})$. Therefore we have $d(\tilde{z}''_n, \tilde{g}_n \circ \tilde{v}(\varphi(j_n))) = d(\tilde{g}_n(\tilde{z}'_n), \tilde{g}_n \circ \tilde{v}(\varphi(i_n))) + o(\epsilon_n^{-1})$. From Lemma 5.3, we get then the equality $d(\tilde{z}''_n, \tilde{g}_n \circ \tilde{v}(\varphi(j_n))) = d(\tilde{z}'_n, \tilde{v}(\varphi(i_n))) + o(\epsilon_n^{-1})$. Using the equality of the paragraph above, we get $d(\tilde{z}''_n, \tilde{g}_n \circ \tilde{v}(\varphi(j_n))) = d(\tilde{z}_n, \tilde{v}(\varphi(j_n)) + o(\epsilon_n^{-1})$. Since almost geodesics $\tilde{J}_n, \tilde{J}'_n$ lie 6ϵ-neighborhood each other, it follows that we have $d(\tilde{z}''_n, \tilde{z}_n) = o(\epsilon_n^{-1})$.

By definition, we have $\tilde{z}''_n = \rho_n(a'_n) \circ \tilde{g}_n \circ \rho_n(a_n^{-1})(\tilde{z}_n)$. Therefore we have

$$d(\tilde{z}_n, \tilde{z}''_n) - d(\tilde{z}''_n, \rho_n(a'_n a_n^{-1})(\tilde{z}_n)) \leq d(\rho_n(a'_n a_n^{-1})(\tilde{z}_n), \tilde{z}_n) \text{ and}$$

$$d(\rho_n(a'_n a_n^{-1})(\tilde{z}_n), \tilde{z}_n) \leq d(\tilde{z}_n, \tilde{z}''_n) + d(\tilde{z}''_n, \rho_n(a'_n a_n^{-1})(\tilde{z}_n)).$$

We have also

$$d(\rho_n((a'_n)^{-1})\tilde{z}''_n, \rho_n(a_n^{-1})\tilde{z}_n) = d(\tilde{g}_n \circ \rho_n(a_n^{-1})(\tilde{z}_n), \rho_n(a_n^{-1})(\tilde{z}_n)) = o(\epsilon_n^{-1}).$$

Thus we finally get the equality $d(\rho_n(a'_n a_n^{-1})(\tilde{z}_n), \tilde{z}_n) = o(\epsilon_n^{-1})$.

We should notice that the $o(\epsilon_n^{-1})$ is "uniform", namely there is a sequence $\delta_n \to 0$ independent of $(\tilde{z}_n)$ such that $d(\rho_n(a'_n a_n^{-1})(\tilde{z}_n), \tilde{z}_n) \leq \delta_n \epsilon_n^{-1}$. □

We shall use this claim to prove the following lemma.

Lemma 5.8. *There is N such that for $n \geq N$, $\rho_n(a_n^{-1} a'_n) = id$.*

We will use the following lemma

Lemma 5.9. *Let $[A_n, B_n] \subset \mathbb{H}^3$ be a sequence of geodesic segments between A_n and B_n such that $l([A_n, B_n])$ is $\Theta(\epsilon_n^{-1})$ and let $\delta_n, \delta'_n \in \pi_1(M)$ be two sequences such that the distances $d(A_n, \rho_n(\delta_n)(A_n))$, $d(A_n, \rho_n(\delta'_n)(A_n))$, $d(B_n, \rho_n(\delta_n)(B_n))$ and $d(B_n, \rho_n(\delta'_n)(B_n))$ are all $o(\epsilon_n^{-1})$. Then there is N such that for $n \geq N$, $[\rho_n(\delta_n), \rho_n(\delta'_n)] = Id$.*

Proof. This comes directly from the arguments that M. Kapovich used in [Ka, Theorem 10.24] to prove that the action is small (cf. [Ka, p. 239]). □

Proof of Lemma 5.8. Set $e = 2p + 1$ where p is the number of the branches of τ^1. If we fix some $n \in \mathbb{N}$, we can find two different integers i_1 and i_2 between $\psi(n)$ and $\psi(n) + e$ such that $bo(i_1) = bo(i_2)$. Let $K_n \subset J_n$ be the segment of s_n with train route $(b(k))_{i_1 \leq k \leq i_2}$, and let $\delta_n \in \pi_1(M)$ be the element such that $\rho_n(\delta_n)$ takes $\tilde{v}(i_1)$ to $\tilde{v}(i_2)$. The isometry $\rho_n(\delta_n)$ acts as a translation on the lift $\tilde{K}_n$ that lies in $\tilde{s}_n$. Since $|i_2 - i_1| \leq e$,we can extract a subsequence such that δ_n does not depend on n. Let us denote it by g.

Let $\tilde{K}_{n-}$ and $\tilde{K}_{n+}$ be the two extremal edges of $\tilde{K}_n$, such that $\rho_n(g)$ maps $\tilde{K}_{n-}$ to $\tilde{K}_{n+}$. By Claim 5.7, up to $o(\epsilon_n^{-1})$, $\rho_n(a_n^{-1} a'_n)$ is the identity on $\tilde{I}_n \cup \tilde{J}_n$. By the same arguments, $\rho_n(a_n'^{-1} a_n)$ is the identity on $\tilde{I}_n \cup \tilde{J}_n$ up to $o(\epsilon_n^{-1})$. It follows that we have :

- for any sequence $\tilde{z}_n \in \mathcal{V}_R(\tilde{K}_{n+})$, $d(\tilde{z}_n, [\rho_n(a_n^{-1} a'_n), \rho_n(g)](\tilde{z}_n)) = o(\epsilon_n^{-1})$;
- for any sequence $\tilde{z}_n \in \mathcal{V}_R(\tilde{K}_{n-})$, $d(\tilde{z}_n, [\rho_n^{-1}(g), \rho_n(a_n^{-1} a'_n)](\tilde{z}_n)) = o(\epsilon_n^{-1})$;
- for any sequence $\tilde{z}_n \in \mathcal{V}_R(\tilde{K}_{n+})$, $d(\tilde{z}_n, [\rho_n(g), \rho_n(a_n'^{-1} a_n)](\tilde{z}_n)) = o(\epsilon_n^{-1})$.

Since $\tilde{K}_{n+}$ and $\tilde{K}_{n-}$ are edges of $\tilde{h}_n(\hat{c}_n^1)$, their lengths are $\Theta(\epsilon_n^{-1})$. Applying Lemma 5.9 to the segments $\tilde{K}_{n-}$ and $\tilde{K}_{n+}$, we see that for sufficiently large n, $[\rho_n(a_n^{-1}a_n'), \rho_n(g)]$, $[\rho_n^{-1}(g), \rho_n(a_n^{-1}a_n')]$ and $[\rho_n(g), \rho_n(a_n'^{-1}a_n)]$ commute with $\rho_n(a_n^{-1}a_n')$. Therefore they belong to an elementary subgroups of $\rho_n(\pi_1(M))$. By [Ka, p. 239], it follows that the group generated by $\rho_n(a_n^{-1}a_n')$ and $\rho_n(g)$ is elementary.

Since $i_1 \neq i_2$, the distance of translation of $\rho_n(g)$ is $\Theta(\epsilon_n^{-1})$.In particular $\rho_n(g)$ is not a parabolic isometry. Since the group generated by $\rho_n(a_n^{-1}a_n')$ and $\rho_n(g)$ is elementary, there are $f \in \pi_1(M)$, $t, t_n \in \mathbb{N}$ such that $g = f^t$ and $a_n^{-1}a_n' = f^{t_n}$. Since the distance of translation of $\rho_n(g)$ is $\Theta(\epsilon_n^{-1})$, the distance of translation of $\rho_n(f)$ is $\Theta(\epsilon_n^{-1})$. But by Lemma 5.7, the distance of translation of $\rho_n(a_n^{-1}a_n')$ is $o(\epsilon_n^{-1})$. Therefore we have $\rho_n(a_n^{-1}a_n') = id$ for sufficiently large n. $\square$

In the construction of a_n and a_n', we chose i_n and j_n so that $g_n \circ h_n(v(\varphi(i_n)))$ and $g_n \circ h_n(v(\varphi(j_n)))$ lie in the same segment ς_n. The consequence of this choice is that the distance measured on $h_n(s_n)$ from $g_n \circ h_n(v(\varphi(i_n)))$ to $g_n \circ h_n(v(\varphi(j_n)))$ is smaller than $R'\frac{\epsilon_n^{-1}}{\psi(n)}$. In fact we only used the fact that the distance measured on $h_n(s_n)$ from $g_n \circ h_n(v(\varphi(i_n)))$ to $g_n \circ h_n(v(\varphi(j_n)))$ is $o(\epsilon_n^{-1})$. Therefore we could have imposed a weaker hypothesis on i_n and j_n, namely that $g_n \circ h_n(v(\varphi(i_n)))$ and $g_n \circ h_n(v(\varphi(j_n)))$ are separated by $o(\psi(n))$-many segments $\varsigma(j, n)$. This allows us to assume that i_n tends to ∞ while the result of Lemma 5.8 that $\rho_n(a_n^{-1}a_n') = id$ still holds for sufficiently large n.

The exhausted reader will probably be glad to know that we shall now construct the homoclinic leaf or the annular lamination of Lemma 5.2.

Let us fix a reference hyperbolic metric on ∂M. Recall that l_+ and l_+' are two half-leaves l_+ and l_+' of L_{rec} whose train routes are $(bo(i))_{i\in\mathbb{N}}$ and $(bo'(i))_{i\in\mathbb{N}}$ respectively. Let $k_n \subset l_+$ and $k_n' \subset l_+'$ be the geodesic arcs with train routes $(bo(i))_{\varphi(i_n)\leq i<\varphi(j_n)}$ and $(bo'(i))_{\varphi(i_n)\leq i<\varphi(j_n)}$ respectively. The endpoints of k_n (resp. k_n') are connected by an arc κ_n (resp. κ_n') which lies in a switch of τ^1. Let e_n (resp. e_n') be the closed geodesic in the free homotopy class of $k_n \cup \kappa_n$ (resp. $k_n' \cup \kappa_n'$). These curves may have some self-intersections. Note that e_n (resp. e_n') represents a_n (resp. a_n').

Since i_n tends to ∞, it follows from the choice of φ (by property (3) of (*)) that the length of κ_n tends to 0 (with respect to our reference metric on ∂M). Moreover, the orientation of $b(\varphi(i_n))$ coincides with the orientation of $b(\varphi(j_n))$, so the curve $k_n \cup \kappa_n$ is getting closer to a geodesic. Let $\mathcal{V}(L_{rec})$ be a neighbourhood of L_{rec}. Since the lengths of κ_n tend to 0 and since $k_n \subset L_{rec}$, the curve $k_n \cup \kappa_n$ lies in $\mathcal{V}(L_{rec})$ for n large enough. So we get that e_n lies in $\mathcal{V}(L_{rec})$ for n large enough. It follows that the sequence (e_n) converges to a connected component of L_{rec} in the Hausdorff topology. The same arguments apply to the sequence (e_n') and shows that it also converges to a connected component of L_{rec} in the Hausdorff topology. Furthermore

it follows also from the choice of k_n and κ_n that we have $i(\lambda, e_n) \longrightarrow 0$ and $i(\lambda, e'_n) \longrightarrow 0$.

By Lemma 5.8,there is an annulus E_n (not necessarily embedded) connecting e_n to e'_n. Notice that since M is irreducible and atoroidal, such an annulus E_n is unique up to homotopy. Since h_n is homotopic to the inclusion, there is an annulus F_n (not necessarily embedded) in $M_n = \mathbb{H}^3/\rho_n(\pi_1(M))$ connecting $h_n(e_n)$ to $h_n(e'_n)$. Furthermore, we can change A_n by a homotopy so that F_n is a subset of A_n.

Assume first that E_n can be homotoped in ∂M. Then e_n is homotopic to e'_n in ∂M. Since e_n and e'_n are closed geodesics with respect to the reference metric on ∂M, they are equal. So E_n is homotopic to e_n. It follows that F_n is homotopic to $h_n(e_n)$. Let $\zeta_n \subset F_n \subset A_n$ be an arc produced in Lemma 5.3. The homotopy between F_n and $h_n(e_n)$ sends ζ_n to an arc $\beta_n \subset h_n(e_n)$. On the other hand ζ_n is an essential arc. Therefore ζ_n is homotopic in A_n to an arc $\zeta'_n \subset \bar{c}_n$ such that $\zeta'_n \not\subset \bar{c}^1_n$. Let us now consider $\zeta'_n \subset \bar{c}_n$ as lying in c_n. By the definition of β_n, there is an arc $\beta'_n \subset h_n^{-1}(\beta_n)$ joining the endpoints of ζ'_n such that $\beta'_n \cup \zeta'_n$ bounds a disc D'_n. Since $\zeta'_n \not\subset \bar{c}^1_n$ and since $\beta_n \subset h_n(\tau_1)$, D'_n can not be homotoped in ∂M. Since ζ'_n lies in $c_n = |\lambda_n|$, we have $i(\partial D'_n, \lambda_n) \leq \int_{\beta'_n} d\lambda_n$. By the Loop Theorem, there is an essential disc D_n such that $i(\partial D_n, \lambda_n) \leq \int_{\beta'_n} d\lambda_n$. In order to get a homoclinic leaf which does not intersect $|\lambda|$ transversely, we are going to show that $i(\lambda_n, \partial D_n) \longrightarrow 0$.

As we have seen above, $\kappa_n \cup k_n$ and $\kappa'_n \cup k'_n$ are close to $e_n = e'_n$. Especially, there is a small arc β''_n joining κ_n to κ'_n with $\int_{\beta''_n} d\lambda_n \longrightarrow 0$. By construction, β'_n and β''_n are homotopic relative to $\kappa_n \cup k_n \cup \kappa'_n \cup k'_n$. Therefore we have $\int_{\beta'_n} d\lambda_n \leq \int_{\beta''_n} d\lambda_n + q_n i(e_n, \lambda_n)$, where q_n is an integer which depends on the number of times that β'_n spirals toward e_n. Let us show that $\beta_n \subsetneq h_n(e_n)$, it will follow that $q_n \leq 1$. By Lemma 5.3, the length of ζ_n in $M_n = \mathbb{H}^3/\rho_n(\pi_1(M))$ is less than 6ϵ. Since $\beta_n \subset h_n(\tau_1)$, it follows from properties $b)$ and $c)$ of Lemma 4.1 that β_n is a quasi-geodesic segment. Therefore, for sufficiently large n, the length of β_n in M_n is less than 7ϵ. By property $b)$ of $h_n(\tau_n)$, the length of $h_n(e_n)$ goes to ∞. Thus we get $\beta_n \subsetneq h_n(e_n)$ and it follows that $q_n \leq 1$. Since we have $\int_{\beta''_n} d\lambda_n \longrightarrow 0$ and $i(\lambda_n, e_n) \longrightarrow 0$, we can conclude that $\int_{\beta'_n} d\lambda_n \longrightarrow 0$. It follows then from the inequality $i(\partial D_n, \lambda_n) \leq \int_{\beta'_n} d\lambda_n$ that $i(\lambda_n, \partial D_n) \longrightarrow 0$. Extract a subsequence such that (∂D_n) converges in the Hausdorff topology to some geodesic lamination H. By Casson's criterion (Lemma 2.4), H contains a homoclinic leaf. Since $i(\lambda_n, \partial D_n) \longrightarrow 0$, H does not intersect $|\lambda|$ transversely.

Assume now that E_n can not be homotoped in ∂M. As we have seen above, $\partial E_n = e_n \cup e'_n$ converges in the Hausdorff topology to a sublamination

E of L_{rec}. Let μ be a measured lamination with $|\mu| = E$. By construction e_n is homotopic to the union of a segment lying in L_{rec} and of a small segment κ_n (whose length with respect to the reference metric tends to 0). It follows that $i(\mu, e_n) \longrightarrow 0$. The same is true for e'_n. By Lemma 2.5 either the lamination E is annular or $S(E)$ contains a homoclinic leaf which does not intersect E transversely. On the other hand, E is a sublamination of L_{rec}. So we have constructed a homoclinic leaf or an annular lamination which does not intersect L_{rec} transversely. Since $|\lambda|$ is a sublamination of L_{rec}, this concludes the proof of Lemma 5.2. □

Now, we can complete the proof of Theorem 2. Let ρ_n, λ_n and λ be as in Theorem 2. If no subsequence of ρ_n converges algebraically, it follows from Lemma 4.1 that ρ_n and λ_n satisfy the assumptions of §5. By Lemma 5.2, there is a homoclinic leaf or an annular lamination which does not intersect $|\lambda|$ transversely. By Lemma 2.7, this contradicts the assumption that $\lambda \in \mathcal{D}(M)$. This completes the proof of Theorem 2. □

6. Conclusion

We shall now deduce Theorem 1 from Theorem 2.

Theorem 1. *Let M be a compact irreducible atoroidal 3-manifold with boundary. Let (m_n) be a sequence in the Teichmüller space $\mathcal{T}(\partial M)$ which converges in the Thurston compactification to a projective lamination $[\lambda]$ contained in $\mathcal{PD}(M)$. Let $q : \mathcal{T}(\partial M) \to CC_0(M)$ be the Ahlfors-Bers map, and suppose that $(\rho_n : \pi_1(M) \to G_n \subset PSL(2, \mathbb{C}))$ is a sequence of discrete faithful representations corresponding to $(q(m_n))$. Then passing to a subsequence, (ρ_n) converges in $AH(N)$.*

Proof. For a simple closed curve $c \subset \partial M$, we denote by $l_{m_n}(c)$ the length of c with respect to the metric m_n and by $l_{\rho_n}(c)$ the length of the closed geodesic of $\mathbb{H}^3/\rho_n(\pi_1(M))$ in the free homotopy class of c. By [Th2] (see also [FLP]), there is a sequence of simple closed curves $c_n \subset \partial M$ whose projective classes converge to $[\lambda]$ in $\mathcal{PML}(\partial M)$ such that $\frac{l_{m_n}(c_n)}{l_{m_0}(c_n)}$ tends to 0 as n goes to infinity.

Using the following result of [BrC], we shall get an upper bound for the sequence $(l_{\rho_n}(\lambda_n))$.

Theorem 2 (Bridgeman-Canary)**.** *For any $Q > 0$, there is a constant $K > 0$ depending only on Q with the following conditions. Let Γ be a finitely generated Kleinian group without torsion such that the shortest meridian length is greater than Q. Let $C(\Gamma)$ be the convex core of $\mathbb{H}^3/\Gamma$, and consider the nearest point retraction $r : \Omega_\Gamma/\Gamma \to \partial C(\Gamma)$. Then r is K-Lipschitz and has a homotopically inverse K-Lipschitz map.*

Let us first verify that we are considering a situation where the hypothesis of this theorem is fulfilled.

Lemma 6.1. *There is a positive number Q such that $l_{m_n}(d) \geq Q$ for any meridian d of ∂M.*

Proof. Assuming the contrary, we have a sequence of meridians (d_n) such that $(l_{m_n}(d_n))$ tends to 0. Let us extract a subsequence so that (d_n) converges with respect to the Hausdorff topology to a geodesic lamination $D \subset M$. By Casson's criterion, D contains a homoclinic leaf. Since $[\lambda] \in \mathcal{PD}(M)$, the lamination D intersects the support of $[\lambda]$ transversely ([Le2, Lemma 3.5], see also [Le1, Lemme 3.3]). It follows that the sequence $i(\lambda, d_n)$ is bounded away from 0. This implies that the sequence $l_{m_n}(d_n)$ tends to ∞. Thus we get a contradiction. $\square$

The length $l_{\rho_n}(c_n)$ is clearly less than the length of any curve in $\partial C(\rho_n(\pi_1(M)))$ which is freely homotopic to c_n. Thus, applying Theorem 2, we get that $\frac{l_{\rho_n}(c_n)}{l_{m_0}(c_n)}$ tends to 0.

Let us denote by λ_n the measured geodesic lamination obtained by endowing c_n with a Dirac measure whose weight is equal to $l_{m_0}(c_n)^{-1}$. The sequence λ_n converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination λ which lies in the projective class $[\lambda]$. Since $\frac{l_{\rho_n}(c_n)}{l_{m_0}(c_n)}$ tends to 0, we have $l_{\rho_n}(\lambda_n) \longrightarrow 0$. Since λ lies in the projective class $[\lambda] \in \mathcal{P}D(M)$, the measured geodesic lamination λ lies in $\mathcal{D}(M)$. Applying Theorem 2, we see that a subsequence of (ρ_n) converges algebraically. $\square$

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Inkang Kim
Department of Mathematics
Seoul National University
Seoul, 151-742, Korea
e-mail: inkang@math.snu.ac.kr

Cyril Lecuire
CNRS
Laboratoire Emile Picard
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse Cedex 4
e-mail: lecuire@math.ups-tlse.fr

Ken'ichi Ohshika
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043, Japan
email: ohshika@math.sci.osaka-u.ac.jp